

# Minimal thinness with respect to subordinate killed Brownian motions

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## Abstract

Minimal thinness is a notion that describes the smallness of a set at a boundary point. In this paper, we provide tests for minimal thinness for a large class of subordinate killed Brownian motions in bounded  $C^{1,1}$  domains,  $C^{1,1}$  domains with compact complements and domains above graphs of bounded  $C^{1,1}$  functions.

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## 1 Introduction

Let  $X = (X_t, \mathbb{P}_x)$  be a Hunt process in an open set  $D \subset \mathbb{R}^d$ ,  $d \geq 2$ . Let  $\partial_M D$  and  $\partial_m D$  be the Martin and minimal Martin boundary of  $D$  with respect to  $X$  respectively. For any  $z \in \partial_M D$ , we denote by  $M^D(x, z)$  the Martin kernel of  $D$  at  $z$  with respect to  $X$ . The family of all excessive functions for  $X$  will be denoted by  $\mathcal{S}$ . For a function  $v : D \rightarrow [0, \infty]$  and a set  $E \subset D$ , the reduced function of  $v$  on  $E$  is defined by  $R_v^E = \inf\{s \in \mathcal{S} : s \geq v \text{ on } E\}$  and its lower semi-continuous regularization is denoted by  $\hat{R}_v^E$ . A set  $E \subset D$  is said to be *minimally thin* in  $D$  at  $z \in \partial_m D$  with respect to  $X$  if  $\hat{R}_{M^D(\cdot, z)}^E \neq M^D(\cdot, z)$ , cf. [14]. A probabilistic interpretation of minimal thinness is given in terms of the process  $X$  conditioned to die at  $z \in \partial_m D$ : For any  $z \in \partial_m D$ , let  $X^z = (X_t^z, \mathbb{P}_x^z)$  denote the  $M^D(\cdot, z)$ -process, Doob's  $h$ -transform of  $X$  with  $h(\cdot) = M^D(\cdot, z)$ . The lifetime of  $X^z$  will be denoted by  $\zeta$ . It is known (see [24]) that  $\lim_{t \uparrow \zeta} X_t^z = z$ ,  $\mathbb{P}_x^z$ -a.s. For  $E \subset D$ , let  $T_E := \inf\{t > 0 : X_t^z \in E\}$ . It is proved in [14, Satz 2.6] that a set  $E \subset D$  is minimally thin at  $z \in \partial_m D$  with respect to  $X$  if and only if there exists  $x \in D$  such that  $\mathbb{P}_x^z(T_E < \zeta) \neq 1$ . This shows that minimal thinness is a concept describing smallness of a set at a boundary point.

The history of minimal thinness goes back to Lelong-Ferrand [25] who introduced this concept in case of the half-space in the setting of classical potential theory. Minimal thinness for general open sets was developed in Naïm [27], while probabilistic interpretation (in terms of Brownian motion) was given by Doob (see e.g. [12]). Various versions of Wiener-type criteria for minimal thinness were developed over the years culminating in the work of Aikawa [2] who, by using the powerful concept of quasi-additivity of capacity, established a criterion for minimal thinness for

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subsets of NTA domains. For a good exposition of these results and methods cf. [3, Part II, 7]. In case of a  $C^{1,1}$  domain  $D \subset \mathbb{R}^d$ , the finite part of the minimal Martin boundary  $\partial_m D$  coincides with the Euclidean boundary  $\partial D$ , and Aikawa's criterion reads as follows: Let  $E$  be a Borel subset of  $D$ . If  $E$  is minimally thin at  $z \in \partial D$ , then

$$\int_{E \cap B(z,1)} |x - z|^{-d} dx < \infty. \quad (1.1)$$

Conversely, if  $E$  is the union of a subfamily of Whitney cubes of  $D$  and (1.1) holds, then  $E$  is minimally thin in  $D$  at  $z$ .

Note that all works listed above pertain to the classical potential theory related to Brownian motion. For more general Hunt processes, although the general theory of minimal thinness was developed by Föllmer already in 1969, see [14], until recently no concrete criteria for minimal thinness were known. The first paper addressing this question was [20] which dealt with minimal thinness of subsets of the half-space for a large class of subordinate Brownian motions. Quite general results for a large class of symmetric Lévy processes in  $\kappa$ -fat open sets were obtained in [23]. The special case of a  $C^{1,1}$  open set  $D$  was given in [23, Corollary 1.5]. We present here a slightly simplified version of the main result of [23]. Assume that  $X$  is an isotropic Lévy process in  $\mathbb{R}^d$ ,  $d \geq 2$ , with characteristic exponent  $\Psi(x) = \Psi(|x|)$  satisfying the following weak scaling condition: There exist constants  $0 < \delta_1 \leq \delta_2 < 1$  and  $a_1, a_2 > 0$  such that

$$a_1 \lambda^{2\delta_1} \Psi(t) \leq \Psi(\lambda t) \leq a_2 \lambda^{2\delta_2} \Psi(t), \quad \lambda \geq 1, t \geq 1. \quad (1.2)$$

We note that many subordinate Brownian motions, particularly all isotropic stable processes, satisfy the above condition. Let  $X^D$  be the process  $X$  killed upon exiting a  $C^{1,1}$  open set  $D$ . If a Borel set  $E \subset D$  is minimally thin in  $D$  at  $z \in \partial D$  with respect to  $X^D$ , then (1.1) holds true. The converse is also true provided  $E$  is the union of a subfamily of Whitney cubes of  $D$ . Thus one obtains the same Aikawa-type criterion for minimal thinness regardless of the particular isotropic Lévy process  $X$  as long as  $X$  satisfies the weak scaling condition (1.2). This is a somewhat surprising result. An explanation for this hinges on sharp two-sided estimates for the Green function of  $X^D$  which imply that the singularity of the Martin kernel  $M^D(x, z)$  near  $z \in \partial D$  is of the order  $|x - z|^{-d}$  for all such processes.

The purpose of this paper is to exhibit a large class of (non-Lévy) Markov processes for which the Aikawa-type criterion for minimal thinness depends on the particular process and is different from (1.1). This class consists of subordinate killed Brownian motions via subordinators having Laplace exponents satisfying a certain weak scaling condition. Let us now precisely formulate the setting and results.

Let  $W = (W_t, \mathbb{P}_x)$  be a Brownian motion in  $\mathbb{R}^d$ ,  $d \geq 2$ , with transition density

$$p(t, x, y) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x - y|^2}{4t}\right), \quad t > 0, x, y \in \mathbb{R}^d.$$

Let  $S = (S_t)_{t \geq 0}$  be an independent subordinator with Laplace exponent  $\phi : (0, \infty) \rightarrow (0, \infty)$ , i.e.,  $\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}$ ,  $t \geq 0$ ,  $\lambda > 0$ . The process  $X = (X_t, \mathbb{P}_x)$  defined by  $X_t = W_{S_t}$ ,  $t \geq 0$ , is called a subordinate Brownian motion. It is an isotropic Lévy process with characteristic exponent  $\Psi(x) = \phi(|x|^2)$ . Let  $D$  be an open subset of  $\mathbb{R}^d$ , and let  $X^D$  be the process  $X$  killed upon exiting  $D$ . This process is known as a killed subordinate Brownian motion. By reversing the order of subordination and killing one obtains a different process. Assume from now on that  $D$  is a domain

(i.e., connected open set) in  $\mathbb{R}^d$ , and let  $W^D = (W_t^D, \mathbb{P}_x)$  be the Brownian motion  $W$  killed upon exiting  $D$ . The process  $Y^D = (Y_t^D, \mathbb{P}_x)$  defined by  $Y_t^D = W_{S_t}^D$ ,  $t \geq 0$ , is called a subordinate killed Brownian motion. It is a Hunt process and its infinitesimal generator is given by  $-\phi(-\Delta|_D)$  where  $\Delta|_D$  is the Dirichlet Laplacian.

Recall that the Laplace exponent of a subordinator is a Bernstein function, i.e., it has the representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \mu(dx),$$

with  $b \geq 0$  and  $\mu$  a measure on  $(0, \infty)$  satisfying  $\int_{(0,\infty)} (1 \wedge x) \mu(dx) < \infty$ , which is called the Lévy measure of  $S$ . The potential measure of the subordinator  $S$  is defined by  $U(A) = \int_0^\infty \mathbb{P}(S_t \in A) dt$ . A Bernstein function  $\phi$  is called a complete Bernstein function if its Lévy measure has a completely monotone density. A Bernstein function  $\phi$  is called a special Bernstein function if the function  $\lambda \mapsto \lambda/\phi(\lambda)$  is also a Bernstein function. The function  $\lambda \mapsto \lambda/\phi(\lambda)$  is called the conjugate Bernstein function of  $\phi$ . It is well known that any complete Bernstein function is a special Bernstein function. For this and other properties of complete and special Bernstein functions, see [28].

In this the paper we will impose following assumptions:

- (A1) the potential measure of  $S$  has a decreasing density  $u$ ;
- (A2) the Lévy measure of  $S$  is infinite and has a decreasing density  $\mu$ ;
- (A3) there exist constants  $\sigma > 0$ ,  $\lambda_0 > 0$  and  $\delta \in (0, 1]$  such that

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \leq \sigma t^{-\delta} \quad \text{for all } t \geq 1 \text{ and } \lambda \geq \lambda_0.$$

Depending on whether our domain  $D$  is bounded or unbounded, we will consider the following two sets of conditions.

- (A4) If  $D$  is bounded and  $d = 2$ , we assume that there are  $\sigma_0 > 0$  and  $\delta_0 \in (0, 2)$  such that

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \geq \sigma_0 t^{-\delta_0} \quad \text{for all } t \geq 1 \text{ and } \lambda \geq \lambda_0.$$

- (A5) If  $D$  is bounded and  $d = 2$ , we assume that

$$\int_0^1 \frac{d\lambda}{\phi(\lambda)} < \infty.$$

- (A6) If  $D$  is unbounded then we assume that  $d \geq 3$  and that there are  $\beta, \sigma_1 > 0$  such that

$$\frac{u(\lambda t)}{u(\lambda)} \geq \sigma_1 t^{-\beta} \quad \text{for all } t \geq 1 \text{ and } \lambda > 0. \tag{1.3}$$

Assumptions (A1)–(A5) were introduced and used in [18] and [19]. It is easy to check that if  $\phi$  is a complete Bernstein function satisfying condition (H1): there exist  $a_1, a_2 > 0$  and  $\delta_1, \delta_2 \in (0, 1)$  satisfying

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1,$$

then **(A1)**–**(A4)** are automatically satisfied. One of the reasons for adopting the more general setup above is to cover the case of geometric stable and iterated geometric stable subordinators. Suppose that  $\alpha \in (0, 2)$  for  $d \geq 2$  and that  $\alpha \in (0, 2]$  for  $d \geq 3$ . A geometric  $(\alpha/2)$ -stable subordinator is a subordinator with Laplace exponent  $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ . Let  $\phi_1(\lambda) := \log(1 + \lambda^{\alpha/2})$ , and for  $n \geq 2$ ,  $\phi_n(\lambda) := \phi_1(\phi_{n-1}(\lambda))$ . A subordinator with Laplace exponent  $\phi_n$  is called an iterated geometric subordinator. It is easy to check that the functions  $\phi$  and  $\phi_n$  satisfy **(A1)**–**(A6)**, but they do not satisfy **(H1)**.

Assumption **(A1)** implies that  $\phi$  is a special Bernstein function, see, for instance, [33, Theorem 5.1]. Moreover, **(A3)** implies  $b = 0$ , **(A2)** implies that  $\mu((0, \infty)) = \infty$ , and **(A5)** is equivalent to the transience of  $X$ . In case  $d \geq 3$ ,  $X$  is always transient.

Condition **(A6)** is only assumed when  $D$  is unbounded and can be restated as

$$\frac{u(R)}{u(r)} \geq \sigma_1 \left( \frac{R}{r} \right)^{-\beta}, \quad 0 < r \leq R < \infty. \quad (1.4)$$

Under **(A1)**–**(A3)**, the inequality in (1.4) is valid with  $\beta = 2 - \delta$  whenever  $0 < r \leq R \leq 1$ , (see (2.11) and (2.12) below). So **(A6)** is mainly a condition about the behavior of  $u$  near infinity. It follows easily from [21] that if  $\phi$  is a complete Bernstein function satisfying, in addition to **(H1)**, also condition **(H2)**: there exist  $a_3, a_4 > 0$  and  $\delta_3, \delta_4 \in (0, 1)$  satisfying

$$a_3 \lambda^{\delta_3} \phi(t) \leq \phi(\lambda t) \leq a_4 \lambda^{\delta_4} \phi(t), \quad \lambda \leq 1, t \leq 1,$$

then **(A6)** is satisfied, see [21, Corollary 2.4]. There are plenty of examples of complete Bernstein functions which satisfy **(A6)** but not **(H2)**. For any  $m > 0$  and  $\alpha \in (0, 2)$ , the function  $\phi(\lambda) := (\lambda + m^{2/\alpha})^{\alpha/2} - m$ , the Laplace exponent of a relativistic stable subordinator, is such an example.

Recall that an open set  $D$  in  $\mathbb{R}^d$  is said to be a (uniform)  $C^{1,1}$  open set if there exist a localization radius  $R > 0$  and a constant  $\Lambda > 0$  such that for every  $z \in \partial D$ , there exist a  $C^{1,1}$ -function  $\psi = \psi_z : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  satisfying  $\psi(0) = 0$ ,  $\nabla \psi(0) = (0, \dots, 0)$ ,  $\|\nabla \psi\|_\infty \leq \Lambda$ ,  $|\nabla \psi(x) - \nabla \psi(w)| \leq \Lambda|x - w|$ , and an orthonormal coordinate system  $CS_z$  with its origin at  $z$  such that

$$B(z, R) \cap D = \{y = (\tilde{y}, y_d) \text{ in } CS_z : |y| < R, y_d > \psi(\tilde{y})\}.$$

The pair  $(R, \Lambda)$  is called the characteristics of the  $C^{1,1}$  open set  $D$ .

Recall that an open set  $D$  is said to satisfy the interior and exterior balls conditions with radius  $R_1$  if for every  $z \in \partial D$ , there exist  $x \in D$  and  $y \in \overline{D}^c$  such that  $\text{dist}(x, \partial D) = R_1$ ,  $\text{dist}(y, \partial D) = R_1$ ,  $B(x, R_1) \subset D$  and  $B(y, R_1) \subset \overline{D}^c$ . It is known, see [4, Definition 2.1 and Lemma 2.2], that an open set  $D$  is a  $C^{1,1}$  open set if and only if it satisfies the interior and exterior ball conditions. By taking  $R$  smaller if necessary, we will always assume a  $C^{1,1}$  open set with characteristics  $(R, \Lambda)$  also satisfies the interior and exterior balls conditions with the same radius  $R$ .

We can now state the main result of this paper. By  $\delta(x)$  we denote the distance of the point  $x \in D$  to the boundary  $\partial D$ .

**Theorem 1.1** *Assume that  $\phi$  is a Bernstein function satisfying **(A1)**–**(A6)**. Let  $D \subset \mathbb{R}^d$  be either a bounded  $C^{1,1}$  domain, or a  $C^{1,1}$  domain with compact complement or a domain above the graph of a bounded  $C^{1,1}$  function.*

(1) *If  $E$  is minimally thin in  $D$  at  $z \in \partial D$  with respect to  $Y^D$ , then*

$$\int_{E \cap B(z, 1)} \frac{\delta(x)^2 \phi(\delta(x)^{-2}) \phi'(|x - z|^{-2})}{|x - z|^{d+4} \phi(|x - z|^{-2})^2} dx < \infty. \quad (1.5)$$

(2) Conversely, if  $E$  is the union of a subfamily of Whitney cubes of  $D$  and (1.5) holds true, then  $E$  is minimally thin in  $D$  at  $z \in \partial D$  with respect to  $Y^D$ .

Since minimal thinness is defined for points in the minimal Martin boundary, the first step in proving this theorem is the identification of the finite part of the (minimal) Martin boundary of  $D$  with its Euclidean boundary. In case of a bounded Lipschitz domain, special subordinator  $S$ , and  $d \geq 3$ , this was accomplished in [31, Theorem 4.3] (see also [33, Theorem 5.84]). The method employed in [31, 33] heavily depended on the fact that the semigroup of the killed Brownian motion  $W^D$  in a bounded Lipschitz domain  $D$  is intrinsically ultracontractive which implies that all excessive functions with respect to  $W^D$  are purely excessive. In fact, [31] proves that there is 1-1 correspondence between the cone of excessive (respectively non-negative harmonic) functions of  $W^D$  and the cone of excessive (respectively non-negative harmonic) functions of  $Y^D$ , thus allowing an easy transfer of many results valid for  $W^D$  to results for  $Y^D$ . In case of an unbounded domain, the semigroup of  $W^D$  is no longer intrinsically ultracontractive and the method from [31] cannot be used to identify the finite part of the (minimal) Martin boundary of  $D$  with its Euclidean boundary.

In the case of killed subordinate Brownian motions, one of the main tools used in identifying the (minimal) Martin boundary of a (possibly) unbounded open set is the boundary Harnack principle.

In the present case of subordinate killed Brownian motions, the boundary Harnack principle is not yet available. As a substitute for the boundary Harnack principle, we first establish sharp two-sided estimates on the Green functions of subordinate killed Brownian motions in any  $C^{1,1}$  domain with compact complement or any domain above the graph of a bounded  $C^{1,1}$  function. This is done in Section 3, see Theorems 3.1 and 3.2. In Section 4, by using some ideas from [31], we then show that the Martin kernel  $M_Y^D(\cdot, \cdot)$  can be extended from  $D \times D$  to  $D \times \overline{D}$ , cf. Proposition 4.4. By using sharp two-sided estimates of the Green function, we subsequently establish in Theorems 4.5 and 4.6 sharp two-sided estimates for the Martin kernel  $M_Y^D(x, z)$ ,  $x \in D$ ,  $z \in \partial D$ . The remaining part of the section is devoted to proving that the finite part of the (minimal) Martin boundary of  $D$  can be identified with its Euclidean boundary in case  $D$  is either a bounded  $C^{1,1}$  domain, a  $C^{1,1}$  domain with compact complement or a domain above the graph of a bounded  $C^{1,1}$  function. We note that in case of a bounded  $C^{1,1}$  domain (and under the assumptions **(A1)**–**(A5)**) this gives an alternative proof of some of the results from [31]. Results of Sections 3 and 4 might be of independent interest.

Having identified the finite part of the (minimal) Martin boundary with the Euclidean boundary, we can follow the method developed by Aikawa, cf. [2] and [3, Part II, 7], which was also used in [23], to prove Theorem 1.1. One of the main ingredients of this method is the quasi-additivity of the capacity related to the process  $Y^D$ , see Proposition 5.9. This depends on the construction of a measure comparable to the capacity which relies on an appropriate Hardy's inequality. The first result on minimal thinness is a criterion given in Proposition 6.2 stating that a subset  $E$  of  $D$  is minimally thin at  $z \in \partial D$  (with respect to  $Y^D$ ) if and only if  $\sum_{n=1}^{\infty} R_{M_Y^D(\cdot, z)}^{E_n}(x_0) < \infty$ ; here  $E_n = E \cap \{x \in D : 2^{-n-1} \leq |x - z| < 2^{-n}\}$  and  $x_0 \in D$  a fixed point. The proof of this general result depends on an inequality relating the Green function and the Martin kernel of  $Y^D$ , cf. Corollary 4.14. The inequality itself hinges on sharp two-sided estimates of the Green function of  $Y^D$  (cf. Theorems 3.1 and 3.2) and sharp two-sided estimates of the Martin kernel (cf. Theorems 4.5 and 4.6). With the quasi-additivity of capacity and the criterion for minimal thinness from Proposition 6.2 in hand, it is rather straightforward to complete the proof of Theorem 1.1.

As an application of Theorem 1.1, we derive an analogue to a criterion in the classical setting for minimal thinness in the half-space  $\mathbb{H}$  of a set below the graph of a Lipschitz function  $f : \mathbb{R}^{d-1} \rightarrow$

$[0, \infty)$ . In the classical case and the case of killed subordinate Brownian motions in the half-space studied in [23], the criterion states that the set  $A = \{(\tilde{x}, x_d) \in \mathbb{H} : 0 < x_d \leq f(\tilde{x})\}$  is minimally thin at 0 if and only if  $\int_{\{|\tilde{x}| < 1\}} f(\tilde{x}) |\tilde{x}|^{-d} d\tilde{x} < \infty$ . For the subordinate killed Brownian motion  $Y^D$  the criterion depends on the underlying Bernstein function  $\phi$  and says that  $A$  is minimally thin at 0 if and only if

$$\int_{\{|\tilde{x}| < 1\}} \frac{f(\tilde{x})^3 \phi(f(\tilde{x})^{-2}) \phi'(|\tilde{x}|^{-2})}{|\tilde{x}|^{d+4} \phi(|\tilde{x}|^{-2})^2} d\tilde{x} < \infty,$$

see Proposition 6.5 and Remark 6.6 for the precise statement.

Finally, we give some examples. We first look at three processes related to the stable process: (1)  $X^D$  – the isotropic  $\alpha$ -stable process killed upon exiting  $D$ , (2)  $Y^D$  – the subordinate killed Brownian motion in  $D$  with  $(\alpha/2)$ -stable subordinator, and (3)  $Z^D$  – the censored  $\alpha$ -stable process in  $D$ . Following [26] we briefly indicate how to prove criteria for minimal thinness for the censored process, and then compare minimal thinness of a given set with respect to these processes and the index of stability  $\alpha$ . Roughly, minimal thinness for  $Z^D$  implies minimal thinness for  $X^D$  which in turn implies minimal thinness for  $Y^D$ , see Corollary 7.3 for the precise statement. We also show that the converse does not hold. At the end of Section 7, we give some examples related to subordinate killed Brownian motions via geometric stable subordinators.

Organization of the paper: In the next section we give some preliminaries on Bernstein functions satisfying conditions **(A1)**–**(A5)** and on the subordinate killed Brownian motion  $Y^D$  and its relation to the killed subordinate Brownian motion. In Section 3 we prove sharp two-sided estimates for the Green function and the jumping kernel of  $Y^D$ . In Section 4 we identify the finite part of the (minimal) Martin boundary with the Euclidean boundary and give sharp two-sided estimates on the Martin kernel of  $Y^D$ . We continue in Section 5 with the proof of the quasi-additivity of the capacity. Results about minimal thinness are proved in Section 6. The paper concludes with criteria for minimal thinness with respect to processes related to the stable case, and with respect to subordinate killed Brownian motions via geometric stable subordinators.

In this paper, we use the letter  $c$ , with or without subscripts, to denote a constant, whose value may change from one appearance to another. The notation  $c(\cdot, \dots, \cdot)$  specifies the dependence of the constant. The dependence of the constants on the domain  $D$  (including the dimension  $d$ ) and the Bernstein function  $\phi$  will not be explicitly mentioned. For any two positive functions  $f$  and  $g$ ,  $f \asymp g$  means that there is a positive constant  $c \geq 1$  so that  $c^{-1}g \leq f \leq cg$  on their common domain of definition. We will use “:=” to denote a definition, which is read as “is defined to be”. For  $a, b \in \mathbb{R}$ ,  $a \wedge b := \min\{a, b\}$  and  $a \vee b := \max\{a, b\}$ .

## 2 Preliminaries

In this section we first collect several properties of Bernstein functions and then collect some results on the subordinate killed Brownian motion  $Y^D$  and its relation to the killed subordinate Brownian motion  $X^D$ .

**Lemma 2.1** (a) *For every Bernstein function  $\phi$ ,*

$$1 \wedge \lambda \leq \frac{\phi(\lambda t)}{\phi(t)} \leq 1 \vee \lambda, \quad \text{for all } t > 0, \lambda > 0. \quad (2.1)$$

(b) *If  $\phi$  is a special Bernstein function, then  $\lambda \mapsto \lambda^2 \phi'(\lambda)$  and  $\lambda \mapsto \lambda^2 \frac{\phi'(\lambda)}{\phi(\lambda)^2}$  are increasing functions. Furthermore, for any  $\gamma > 2$ ,  $\lim_{\lambda \rightarrow 0} \lambda^\gamma \frac{\phi'(\lambda)}{\phi(\lambda)^2} = 0$ .*

(c) If  $\phi$  is a special Bernstein function, then for every  $d \geq 2$ ,  $\gamma \geq 2$ ,  $\lambda > 0$ ,  $b \in (0, 1]$  and  $a \in [1, \infty)$  it holds that

$$\frac{b}{a^{d+\gamma+1}\lambda^{d+\gamma}} \frac{\phi'(\lambda^{-2})}{\phi(\lambda^{-2})^2} \leq \frac{1}{t^{d+\gamma}} \frac{\phi'(t^{-2})}{\phi(t^{-2})^2} \leq \frac{a}{b^{d+\gamma+1}\lambda^{d+\gamma}} \frac{\phi'(\lambda^{-2})}{\phi(\lambda^{-2})^2}, \quad \text{for all } t \in [b\lambda, a\lambda]. \quad (2.2)$$

Part (a) is well known, part (b) is proved in [18, Lemma 4.1], and part (c) can be proved in the same way as [19, Corollary 2.2] where the proof is given for  $\gamma = 2$ . We will frequently use all three properties of the lemma, often without explicitly mentioning it.

Let  $W$  be a Brownian motion in  $\mathbb{R}^d$ ,  $D \subset \mathbb{R}^d$  a domain, and  $W^D$  a Brownian motion killed upon exiting  $D$ . We denote by  $p^D(t, x, y)$ ,  $t > 0$ ,  $x, y \in D$ , the transition densities of  $W^D$ , and by  $(P_t^D)_{t \geq 0}$  the corresponding semigroup. Let  $S$  be a subordinator independent of the Brownian motion  $W$ . Let  $Y_t^D = W_{S_t}^D$  be the corresponding subordinate killed Brownian motion in  $D$ . The process  $Y^D$  is a symmetric Hunt process, cf. [32]. We will use  $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$  to denote the Dirichlet form associated with  $Y^D$ . The killing measure of  $\mathcal{E}^D$  has a density  $\kappa_D$  given by the formula

$$\kappa_D(x) = \int_{(0, \infty)} (1 - P_t^D 1(x)) \mu(dt), \quad x \in D. \quad (2.3)$$

It follows from the general theory of Dirichlet forms that for every  $v \in \mathcal{D}(\mathcal{E}^D)$  it holds that

$$\mathcal{E}^D(v, v) \geq \int_D v(x)^2 \kappa_D(x) dx. \quad (2.4)$$

Let  $(R_t^D)_{t \geq 0}$  be the transition semigroup of  $Y^D$ . We will need to compare this semigroup with the semigroup of the killed subordinate Brownian motion. Recall that  $X_t = W_{S_t}$  is the subordinate Brownian motion and  $(X_t^D)_{t \geq 0}$  is the subprocess of  $X$  killed upon exiting  $D$ . Let  $(Q_t^D)_{t \geq 0}$  denote the transition semigroup of  $X^D$ . It is well known, cf. [32, Proposition 3.1], that  $(R_t^D)_{t \geq 0}$  is subordinate to  $(Q_t^D)_{t \geq 0}$  in the sense that

$$R_t^D f(x) \leq Q_t^D f(x) \quad \text{for all Borel } f : D \rightarrow [0, \infty) \text{ all } t \geq 0 \text{ and all } x \in D. \quad (2.5)$$

Let  $j_X(x)$  denote the density of the Lévy measure of the process  $X$ . Then

$$j_X(x) = \int_{(0, \infty)} p(t, x, 0) \mu(dt) = \int_{(0, \infty)} (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \mu(dt).$$

Clearly,  $j_X$  is a continuous function of  $x$  on  $\mathbb{R}^d \setminus \{0\}$  and radial (that is,  $j_X(x) = j_X(|x|)$ ). Let  $\kappa_D^X$  denote the killing function of  $X^D$ . Then

$$\kappa_D^X(x) = \int_{D^c} j_X(x - y) dy, \quad x \in D, \quad (2.6)$$

and  $\kappa_D^X$  is a continuous function of  $x \in D$ .

**Lemma 2.2** *For any open set  $D \subset \mathbb{R}^d$ ,*

$$\kappa_D^X(x) \leq \kappa_D(x), \quad \text{for almost all } x \in D. \quad (2.7)$$

**Proof.** Using (2.5), the Lemma follows from the argument of [30, Proposition 3.2].  $\square$

Assume  $\phi$  is a Bernstein function satisfying **(A1)** so that the potential measure of  $S$  has a decreasing density  $u(t)$ . Then the Green function of the subordinate killed Brownian motion  $Y^D$ , denoted by  $U^D(x, y)$ ,  $x, y \in D$ , is given by the formula

$$U^D(x, y) = \int_0^\infty p^D(t, x, y)u(t) dt = \int_0^\infty r^D(t, x, y) dt, \quad x, y \in D. \quad (2.8)$$

Similarly, the Green function of  $X$ , denoted by  $G_X(x, y)$ ,  $x, y \in \mathbb{R}^d$ , is given by

$$G_X(x, y) = \int_0^\infty p(t, x, y)u(t) dt, \quad x, y \in \mathbb{R}^d. \quad (2.9)$$

Since  $p^D(t, x, y) \leq p(t, x, y)$  for all  $x, y \in D$ , we see from (2.8) and (2.9) that

$$U^D(x, y) \leq G_X(x, y), \quad \text{for all } x, y \in D. \quad (2.10)$$

Assume now that  $\phi$  is a Bernstein function satisfying **(A1)**–**(A5)** and let  $S$  be a subordinator with Laplace exponent  $\phi$ . The potential density  $u(t)$  of  $S$  satisfies the following two estimates:

$$u(t) \leq (1 - 2e^{-1})^{-1} \frac{\phi'(t^{-1})}{t^2 \phi(t^{-1})^2}, \quad t > 0, \quad (2.11)$$

and, for every  $M > 0$  there exists  $c_1 = c_1(M) > 0$  such that

$$u(t) \geq c_1 \frac{\phi'(t^{-1})}{t^2 \phi(t^{-1})^2}, \quad 0 < t \leq M. \quad (2.12)$$

For the upper estimate see [18, Lemma A.1], and for the lower [18, Proposition 3.4]

The density  $\mu(t)$  of the Lévy measure of  $S$  satisfies the following two estimates:

$$\mu(t) \leq (1 - 2e^{-1})^{-1} t^{-2} \phi'(t^{-1}), \quad t > 0, \quad (2.13)$$

and, for every  $M > 0$  there exists  $c_2 = c_2(M) > 0$  such that

$$\mu(t) \geq c_2 t^{-2} \phi'(t^{-1}), \quad 0 < t \leq M. \quad (2.14)$$

For the upper estimate see [18, Lemma A.1], and for the lower [18, Proposition 3.3].

Recall that  $G_X(x, y)$  denotes the Green function of the subordinate Brownian motion  $X_t = W_{S_t}$ . When  $d \geq 3$  we have that there exists  $c_3 > 0$  such that

$$G_X(x, y) \leq c_3 \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2} \phi(|x - y|^{-2})^2}, \quad x, y \in \mathbb{R}^d. \quad (2.15)$$

This can be proved by following the proof of [21, Lemma 3.2(b)] using (2.11) and [18, Lemma 4.1]. Moreover, by [18, Proposition 4.5] we have the following two-sided inequality: For every  $d \geq 2$  and  $M > 0$ , there exists  $c_4 = c_4(M) > 1$  such that

$$c_4^{-1} \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2} \phi(|x - y|^{-2})^2} \leq G_X(x, y) \leq c_4 \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2} \phi(|x - y|^{-2})^2}, \quad |x - y| \leq M. \quad (2.16)$$

The Lévy density of  $X$  also has the following two-sided estimates by [18, Proposition 4.2]: For every  $M > 0$  there exists  $c_5 = c_5(M) > 0$  such that

$$c_5^{-1} r^{-d-2} \phi'(r^{-2}) \leq j_X(r) \leq c_5 r^{-d-2} \phi'(r^{-2}), \quad r \in (0, M]. \quad (2.17)$$

Thus, by using Lemma 2.1(a) and (c), for every  $M > 0$ ,

$$j_X(r) \leq c j_X(2r), \quad r \in (0, M]. \quad (2.18)$$



### 3 Kernel estimates on subordinate killed Brownian motion

In this section we assume that  $D \subset \mathbb{R}^d$  is either a bounded  $C^{1,1}$  domain, or a  $C^{1,1}$  domain with compact complement or a domain above the graph of a bounded  $C^{1,1}$  function. We assume that the  $C^{1,1}$  characteristics of  $D$  is  $(R, \Lambda)$ .

Recall that  $(P_t^D)_{t \geq 0}$  denotes the transition semigroup of the killed Brownian motion  $W^D$  and  $p^D(t, x, y)$ ,  $t > 0$ ,  $x, y \in D$ , is the corresponding transition density. It is known that  $p^D(t, x, y)$  satisfies the following short-time estimates (cf. [35, 36, 29]): For any  $T > 0$ , there exist positive constants  $c_1, c_2, c_3, c_4$  such that for any  $t \in (0, T]$  and any  $x, y \in D$ ,

$$p^D(t, x, y) \leq c_1 \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_2|x-y|^2}{t} \right), \quad (3.1)$$

$$p^D(t, x, y) \geq c_3 \left( \frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_4|x-y|^2}{t} \right). \quad (3.2)$$

Thus, by the semigroup property and (3.1), we get there exist positive constants  $c_5, c_6, c_7, c_8$  such that for every  $t > 3$

$$\begin{aligned} p^D(t, x, y) &= \int_D \int_D p^D(1, x, z) p^D(t-2, z, w) p^D(1, w, y) dz dw \\ &\leq c_5 (\delta(x) \wedge 1) (\delta(y) \wedge 1) \\ &\quad \times \int_D \int_D \exp(-c_6|x-z|^2) (t-2)^{-d/2} \exp \left( -\frac{c_6|z-w|^2}{t-2} \right) \exp(-c_6|w-y|^2) dz dw \\ &\leq c_5 (\delta(x) \wedge 1) (\delta(y) \wedge 1) \\ &\quad \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(-c_6|x-z|^2) (t-2)^{-d/2} \exp \left( -\frac{c_6|z-w|^2}{t-2} \right) \exp(-c_6|w-y|^2) dz dw \\ &\leq c_7 (\delta(x) \wedge 1) (\delta(y) \wedge 1) t^{-d/2} \exp \left( -\frac{c_8|x-y|^2}{t} \right). \end{aligned}$$

Combining this with (3.1), we have that there exist positive constant  $c_9, c_{10}$  such that for all  $t > 0$  and any  $x, y \in D$ ,

$$p^D(t, x, y) \leq c_9 \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_{10}|x-y|^2}{t} \right). \quad (3.3)$$

We will use the following bound several times: By the change of variables  $s = c|x-y|^2/t$ , for every  $c > 0$  and  $a \in \mathbb{R}$ , we have

$$\begin{aligned} &\int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-a/2} \exp \left( -\frac{c|x-y|^2}{t} \right) dt \\ &= \int_c^\infty \left( \frac{\sqrt{s/c} \delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\sqrt{s/c} \delta(y)}{|x-y|} \wedge 1 \right) \left( \frac{c|x-y|^2}{s} \right)^{-a/2} e^{-s} \frac{c|x-y|^2}{s^2} ds \\ &\geq c^{1-(a/2)} \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y|} \wedge 1 \right) |x-y|^{-a+2} \int_c^\infty s^{a/2-2} e^{-s} ds. \end{aligned} \quad (3.4)$$

Our first goal is to obtain sharp two-sided estimates on  $U^D$ . Under stronger assumptions on the Laplace exponent  $\phi$  such estimates were given in [33, Theorem 5.91] for bounded  $D$ . In the remainder of this section  $\phi$  is a Bernstein function satisfying **(A1)**–**(A5)**. We first consider the case  $|x-y| \leq M$ .

**Theorem 3.1** *For every  $M > 0$ , there exists a constant  $c = c(M) \geq 1$  such that for all  $x, y \in D$  with  $|x - y| \leq M$ ,*

$$\begin{aligned} & c^{-1} \left( \frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2} \\ & \leq U^D(x, y) \leq c \left( \frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}. \end{aligned} \quad (3.5)$$

**Proof.** *Upper bound:* It follows from (2.10) and (2.16) that there exists a constant  $c_1 > 0$  such that for all  $x, y \in D$  with  $|x - y| \leq M$ ,

$$U^D(x, y) \leq G_X(x, y) \leq c_1 \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}. \quad (3.6)$$

Let  $c_2$  be the constant  $c_{10}$  in (3.3). Since  $t \rightarrow \frac{\phi'(t^{-1})}{\phi(t^{-1})^2}$  is increasing, using (2.11) we have that for  $r > 0$ ,

$$\begin{aligned} I_1(r) &:= \int_0^{r^2} t^{-d/2-1} \exp\left(-\frac{c_2 r^2}{t}\right) u(t) dt \leq c_3 \int_0^{r^2} t^{-d/2-1} \exp\left(-\frac{c_2 r^2}{t}\right) t^{-2} \frac{\phi'(t^{-1})}{\phi(t^{-1})^2} dt \\ &\leq c_3 \frac{\phi'(r^{-2})}{\phi(r^{-2})^2} \int_0^{r^2} t^{-\frac{d}{2}-3} \exp\left(-\frac{c_2 r^2}{t}\right) dt = c_4 r^{-d-4} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2} \int_{c_2}^{\infty} t^{\frac{d}{2}+1} e^{-t} dt. \end{aligned} \quad (3.7)$$

On the other hand, since  $u$  is decreasing, using (2.11) we have that for  $r > 0$ ,

$$\begin{aligned} I_2(r) &:= \int_{r^2}^{\infty} t^{-d/2-1} u(t) dt \leq u(r^2) \int_{r^2}^{\infty} t^{-d/2-1} dt \\ &\leq c_5 r^{-4} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2} \int_{r^2}^{\infty} t^{-d/2-1} dt \leq c_6 r^{-d-4} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2}. \end{aligned} \quad (3.8)$$

It follows from [18, Lemma 4.4] that

$$L := \int_{(2M)^2}^{\infty} t^{-d/2} u(t) dt < \infty. \quad (3.9)$$

Thus from (2.8), (3.3) and (3.7)–(3.9), we have that, for  $|x - y| \leq M$ ,

$$\begin{aligned} U^D(x, y) &= \int_0^{\infty} p^D(t, x, y) u(t) dt \\ &\leq \int_0^{|x-y|^2} p^D(t, x, y) u(t) dt + \int_{|x-y|^2}^{(2M)^2} p^D(t, x, y) u(t) dt + \int_{(2M)^2}^{\infty} p^D(t, x, y) u(t) dt \\ &\leq c_7 \int_0^{|x-y|^2} t^{-d/2-1} \delta(x)\delta(y) \exp\left(-\frac{c_2 |x-y|^2}{t}\right) u(t) dt \\ &\quad + c_7 \int_{|x-y|^2}^{(2M)^2} t^{-d/2-1} \delta(x)\delta(y) u(t) dt + c_7 \int_{(2M)^2}^{\infty} t^{-d/2} \delta(x)\delta(y) u(t) dt \\ &\leq c_7 \delta(x)\delta(y) (I_1(|x-y|) + I_2(|x-y|) + L) \leq c_8 \frac{\delta(x)\delta(y)}{|x-y|^2} \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}. \end{aligned}$$

In the last inequality above we use the fact that  $r \rightarrow r^{-d-4} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2}$  is a decreasing function and is thus bounded from below by a positive constant on  $(0, M^2]$ . Together with (3.6) this gives the upper bound in (3.5).

*Lower bound:* Since  $u$  is decreasing and  $|x - y| \leq M$ , by (3.2) and (2.12),

$$\begin{aligned} U^D(x, y) &\geq c_9 \int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_{10}|x-y|^2}{t} \right) u(t) dt \\ &\geq c_9 u(|x-y|^2) \int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_{10}|x-y|^2}{t} \right) dt \\ &\geq c_{11} \frac{\phi'(|x-y|^{-2})}{|x-y|^4 \phi(|x-y|^{-2})^2} \int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_{10}|x-y|^2}{t} \right) dt. \end{aligned}$$

By combining this with (3.4) we arrive at

$$\begin{aligned} U^D(x, y) &\geq c_{12} \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y|} \wedge 1 \right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2} \phi(|x-y|^{-2})^2} \\ &\asymp \left( \frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2} \phi(|x-y|^{-2})^2}. \end{aligned}$$

□

We now assume  $d \geq 3$  and consider our two types of unbounded  $C^{1,1}$  domains and give different estimates for  $U^D$ .

If  $D \subset \mathbb{R}^d$  is a domain above the graph of a bounded  $C^{1,1}$  function, then it follows from [35, 29] that there exist positive constants  $c_1, c_2, c_3$  and  $c_4$  such that for any  $t \in (0, \infty)$  and any  $x, y \in D$ ,

$$p^D(t, x, y) \leq c_1 \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_2|x-y|^2}{t} \right), \quad (3.10)$$

$$p^D(t, x, y) \geq c_3 \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_4|x-y|^2}{t} \right). \quad (3.11)$$

Clearly for  $a > 2$ ,

$$\int_{|x-y|^2}^{\infty} \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-a/2} dt \leq \frac{2}{a-2} \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y|} \wedge 1 \right) \frac{1}{|x-y|^{a-2}}. \quad (3.12)$$

By the change of variables  $s = |x-y|^2/t$  and the inequality

$$\left( \frac{\sqrt{s}\delta(x)}{|x-y|} \wedge 1 \right) \leq \sqrt{s} \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right), \quad s \geq 1,$$

it is easy to see that for  $a \in \mathbb{R}$  and  $b > 0$ , there exist a constant  $c = c(a, b) > 0$  such that

$$\begin{aligned} \int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t}} \wedge 1 \right) t^{-a/2} \exp \left( -\frac{b|x-y|^2}{t} \right) dt \\ \leq c \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y|} \wedge 1 \right) \frac{1}{|x-y|^{a-2}}. \end{aligned} \quad (3.13)$$

If  $D \subset \mathbb{R}^d$  is a  $C^{1,1}$  domain with compact complement, then it follows from [36] that there exist positive constants  $c_5, c_6, c_7$  and  $c_8$  such that for any  $t \in (0, \infty)$  and any  $x, y \in D$ ,

$$p^D(t, x, y) \leq c_5 \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_6|x-y|^2}{t} \right), \quad (3.14)$$

$$p^D(t, x, y) \geq c_7 \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_8|x-y|^2}{t} \right). \quad (3.15)$$

Clearly for  $a > 2$ ,

$$\begin{aligned} & \int_{|x-y|^2}^{\infty} \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-a/2} dt \\ & \leq \frac{2}{a-2} \left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y| \wedge 1} \wedge 1 \right) \frac{1}{|x-y|^{a-2}}. \end{aligned} \quad (3.16)$$

By the change of variables  $s = |x-y|^2/t$  and the inequalities

$$\left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right) \leq \left( \frac{\delta(x)}{(|x-y|/\sqrt{s}) \wedge 1} \wedge 1 \right) \leq \sqrt{s} \left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right), \quad s \geq 1,$$

it is easy to see that for  $a \in \mathbb{R}$  and  $b > 0$ , there exists a constant  $c = c(a, b) > 0$  such that

$$\begin{aligned} & \int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-a/2} \exp \left( -\frac{b|x-y|^2}{t} \right) dt \\ & \leq c \left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y| \wedge 1} \wedge 1 \right) \frac{1}{|x-y|^{a-2}} \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} & \int_0^{|x-y|^2} \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-a/2} \exp \left( -\frac{b|x-y|^2}{t} \right) dt \\ & \geq c^{-1} \left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y| \wedge 1} \wedge 1 \right) \frac{1}{|x-y|^{a-2}}. \end{aligned} \quad (3.18)$$

**Theorem 3.2** Suppose that  $d \geq 3$  and that  $\phi$  is a Bernstein function satisfying **(A1)**–**(A3)** and **(A6)**. (1) Let  $D \subset \mathbb{R}^d$  be a domain above the graph of a bounded  $C^{1,1}$  function. There exists a constant  $c_1 \geq 1$  such that for all  $x, y \in D$ ,

$$\begin{aligned} & c_1^{-1} \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y|} \wedge 1 \right) \frac{u(|x-y|^2)}{|x-y|^{d-2}} \leq U^D(x, y) \\ & \leq c_1 \left( \frac{\delta(x)}{|x-y|} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y|} \wedge 1 \right) \frac{u(|x-y|^2)}{|x-y|^{d-2}}. \end{aligned}$$

(2) Let  $D \subset \mathbb{R}^d$  be a  $C^{1,1}$  domain with compact complement. There exists a constant  $c_1 \geq 1$  such that for all  $x, y \in D$ ,

$$\begin{aligned} & c_1^{-1} \left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y| \wedge 1} \wedge 1 \right) \frac{u(|x-y|^2)}{|x-y|^{d-2}} \leq U^D(x, y) \\ & \leq c_1 \left( \frac{\delta(x)}{|x-y| \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{|x-y| \wedge 1} \wedge 1 \right) \frac{u(|x-y|^2)}{|x-y|^{d-2}}. \end{aligned}$$

**Proof.** We give the proof of (2) first.

*Upper bound:* Using (1.3) and the fact  $u$  is decreasing, we have from (3.14) that

$$\begin{aligned}
U^D(x, y) &= \int_0^\infty p^D(t, x, y) u(t) dt \\
&\leq c_1 \int_0^\infty \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_2 |x - y|^2}{t} \right) u(t) dt \\
&\leq c_3 |x - y|^{2\beta} u(|x - y|^2) \int_0^{|x - y|^2} \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-\beta - d/2} \exp \left( -c_2 \frac{|x - y|^2}{t} \right) dt \\
&\quad + c_1 u(|x - y|^2) \int_{|x - y|^2}^\infty \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} dt.
\end{aligned}$$

Together with (3.16)–(3.17) we obtain the upper bound.

*Lower bound:* Since  $u$  is decreasing, by (3.15)

$$\begin{aligned}
U^D(x, y) &\geq c_4 \int_0^{|x - y|^2} \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_5 |x - y|^2}{t} \right) u(t) dt \\
&\geq c_4 u(|x - y|^2) \int_0^{|x - y|^2} \left( \frac{\delta(x)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{\sqrt{t} \wedge 1} \wedge 1 \right) t^{-d/2} \exp \left( -\frac{c_5 |x - y|^2}{t} \right) dt. \quad (3.19)
\end{aligned}$$

Combining (3.19) and (3.18) we arrive at

$$U^D(x, y) \geq c_6 \left( \frac{\delta(x)}{|x - y| \wedge 1} \wedge 1 \right) \left( \frac{\delta(y)}{|x - y| \wedge 1} \wedge 1 \right) \frac{u(|x - y|^2)}{|x - y|^{d-2}}.$$

Using (3.4) and (3.10)–(3.13), instead of (3.14)–(3.19), the proof of (1) is similar to (2).  $\square$

**Proposition 3.3** *The Green function  $U^D$  is jointly continuous in the extended sense, hence jointly lower semi-continuous, on  $D \times D$ .*

**Proof.** Let  $x, y \in D$ ,  $x \neq y$ , and set  $\eta = |x - y|/2$ . Let  $(x_n, y_n)_{n \geq 1}$  be a sequence in  $D \times D$  converging to  $(x, y)$  and assume that  $|x_n - y_n| \geq \eta$ . For every  $t > 0$ ,  $\lim_{n \rightarrow \infty} p^D(t, x_n, y_n) = p^D(t, x, y)$ . Moreover

$$p^D(t, x_n, y_n) \leq (4\pi t)^{-d/2} \exp \left( -\frac{|x_n - y_n|^2}{4t} \right) \leq (4\pi t)^{-d/2} \exp \left( -\frac{\eta^2}{4t} \right).$$

Since the process  $X$  is transient, we have that

$$\int_0^\infty (4\pi t)^{-d/2} \exp \left( -\frac{\eta^2}{4t} \right) u(t) dt < \infty.$$

Now it follows from the bounded convergence theorem that

$$\lim_{n \rightarrow \infty} U^D(x_n, y_n) = \lim_{n \rightarrow \infty} \int_0^\infty p^D(t, x_n, y_n) u(t) dt = \int_0^\infty p^D(t, x, y) u(t) dt = U^D(x, y).$$

On the other hand, from Theorem 3.1 we get that

$$\lim_{(x_n, y_n) \rightarrow (x, x)} U^D(x_n, y_n) = +\infty = U^D(x, x).$$

Thus  $U^D$  is jointly continuous in the extended sense, and therefore jointly lower semi-continuous.  $\square$

We now recall a result from analysis (see [34, Theorem 1, p. 167]): Any open set  $D \subset \mathbb{R}^d$  is the union of a family  $\{Q_j\}_{j \in \mathbb{N}}$  of closed cubes, with sides all parallel to the axes, satisfying the following properties: (i)  $\text{int}(Q_j) \cap \text{int}(Q_k) = \emptyset$ ,  $j \neq k$ ; (ii) for any  $j$ ,  $\text{diam}(Q_j) \leq \text{dist}(Q_j, \partial D) \leq 4\text{diam}(Q_j)$ , where  $\text{dist}(Q_j, \partial D)$  denotes the Euclidean distance between  $Q_j$  and  $\partial D$ . The family  $\{Q_j\}_{j \in \mathbb{N}}$  above is called a Whitney decomposition of  $D$  and the  $Q_j$ 's are called Whitney cubes (of  $D$ ). We will use  $x_j$  to denote the center of the cube  $Q_j$ . For each cube  $Q_j$  let  $Q_j^*$  denote the interior of the double of  $Q_j$ .

**Corollary 3.4** (i) For every  $M > 0$  there exists a constant  $c_1 = c_1(M) \geq 1$  such that for all Whitney cubes  $Q_j$  whose diameter is less than  $M$ ,

$$c_1^{-1}U^D(x', y) \leq U^D(x, y) \leq c_1U^D(x', y), \quad (3.20)$$

for all  $x, x' \in Q_j$  and all  $y \in D \setminus Q_j^*$  with  $\text{dist}(y, Q_j) < M$ .

(ii) For every  $M > 0$  there exists a constant  $c_2 = c_2(M) > 0$  such that for all cubes  $Q_j$  whose diameter is less than  $M$  and all  $x, x' \in Q_j$ , it holds that

$$U^D(x, x') \geq c_2 G_X(x, x'). \quad (3.21)$$

**Proof.** (i) From the geometry of Whitney cubes it is easy to see that there exists a constant  $c \geq 1$  such that for every cube  $Q_j$  it holds that

$$\begin{aligned} c^{-1}\delta(x) &\leq \delta(x_j) \leq c\delta(x), \quad \text{for all } x \in Q_j, \\ c^{-1}|x - y| &\leq |x_j - y| \leq c|x - y|, \quad \text{for all } x \in Q_j \text{ and all } y \in D \setminus Q_j^*. \end{aligned}$$

Together with Theorem 3.1 and Lemma 2.1(c), these estimates imply that

$$U^D(x, y) \asymp U^D(x_j, y), \quad \text{for all } x \in Q_j \text{ and all } y \in D \setminus Q_j^* \text{ with } \text{dist}(y, Q_j) < M,$$

with a constant independent of  $Q_j$ . This clearly implies the statement of the corollary.

(ii) If  $x, x' \in Q_j$ , then  $|x - x'| \leq \text{diam}(Q_j) \leq \text{dist}(Q_j, \partial D) \leq \delta(x) \wedge \delta(x') \wedge (4M)$ . Thus it follows from (3.5) and (2.16) that

$$U^D(x, x') \geq c_1 \frac{\phi'(|x - x'|^{-2})}{|x - x'|^{d+2} \phi(|x - x'|^{-2})^2} \geq c_2 G_X(x, x').$$

$\square$

Let  $J^D(x, y)$  be the jumping density of  $Y^D$  defined by

$$J^D(x, y) = \int_0^\infty p^D(t, x, y) \mu(t) dt.$$

Clearly  $J^D(x, y) \leq j_X(|x - y|)$ ,  $x, y \in D$ .

Using (2.13), (2.14), (2.17) and the fact that  $t^2\phi'(t)$  is increasing (see Lemma 2.1(b)), the proof of the next proposition is very similar to that of Theorem 3.1.

**Proposition 3.5** *For every  $M > 0$ , there exists a constant  $c = c(M) \geq 1$  such that for all  $x, y \in D$  with  $|x - y| \leq M$ ,*

$$c^{-1} \left( \frac{\delta(x)\delta(y)}{|x - y|^2} \wedge 1 \right) \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2}} \leq J^D(x, y) \leq c \left( \frac{\delta(x)\delta(y)}{|x - y|^2} \wedge 1 \right) \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2}}.$$

For any open subset  $B$  of  $D$ , let  $U^{D,B}(x, y)$  be the Green function of  $Y^D$  killed upon exiting  $B$ . We define the Poisson kernel

$$K^{D,B}(x, y) := \int_B U^{D,B}(x, z) J^D(z, y) dz, \quad (x, y) \in B \times (D \setminus \overline{B}). \quad (3.22)$$

Using the Lévy system for  $Y^D$ , we know that for every open subset  $B$  of  $D$  and every  $f \geq 0$  on  $D \setminus \overline{B}$  and  $x \in B$ ,

$$\mathbb{E}_x [f(Y_{\tau_B}^D); Y_{\tau_B}^D \neq Y_{\tau_B}^D] = \int_{D \setminus \overline{B}} K^{D,B}(x, y) f(y) dy. \quad (3.23)$$

**Lemma 3.6** *For every  $M > 0$ , there exists  $c = c(M) > 0$  such that for any ball  $B(x_0, r) \subset D$  of radius  $r \in (0, 1]$ , we have for all  $(x, y) \in B(x_0, r) \times (D \setminus \overline{B(x_0, r)})$  with  $|x - y| \leq M$ ,*

$$K^{D,B(x_0,r)}(x, y) \leq c \delta(y) \frac{\phi'(|y - x_0| - r)^{-2}}{(|y - x_0| - r)^{d+3}} \phi(r^{-2})^{-1}. \quad (3.24)$$

**Proof.** Let  $B = B(x_0, r)$ . Since  $U^{D,B}(x, y) \leq G_X(x, y)$ , (3.22) and Proposition 3.5 imply that for every  $(x, y) \in B \times (D \setminus \overline{B})$  with  $|x - y| \leq M$ ,

$$\begin{aligned} K^{D,B}(x, y) &\leq \int_B G_X(x, z) J^D(z, y) dz \\ &\leq c_1(M) \int_B G_X(x, z) \left( \frac{\delta(z)\delta(y)}{|z - y|^2} \wedge 1 \right) \frac{\phi'(|z - y|^{-2})}{|z - y|^{d+2}} dz \\ &\leq c_1(M) \delta(y) \int_B G_X(x, z) \frac{\phi'(|z - y|^{-2})}{|z - y|^{d+3}} dz. \end{aligned} \quad (3.25)$$

Since  $|z - y| \geq |y - x_0| - r$  and  $t \rightarrow t^{-d-3}\phi'(t^{-2})$  is decreasing (see Lemma 2.1(b)),

$$\begin{aligned} \int_B G_X(x, z) \frac{\phi'(|z - y|^{-2})}{|z - y|^{d+3}} dz &\leq \frac{\phi'(|y - x_0| - r)^{-2}}{(|y - x_0| - r)^{d+3}} \int_B G_X(x, z) dz \\ &\leq \frac{\phi'(|y - x_0| - r)^{-2}}{(|y - x_0| - r)^{d+3}} \int_{B(0, 2r)} G_X(0, z) dz. \end{aligned} \quad (3.26)$$

By (2.16), we have

$$\begin{aligned} \int_{B(0, 2r)} G_X(0, z) dz &\leq c_2 \int_{B(0, 2r)} |z|^{-d-2} \frac{\phi'(|z|^{-2})}{\phi(|z|^{-2})^2} dz = c_2 \int_0^{2r} r^{-3} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2} dr \\ &\leq 2^{-1} c_3 \phi(2^{-1} r^{-2})^{-1} \leq 2c_4 \phi(r^{-2})^{-1}. \end{aligned} \quad (3.27)$$

Combining (3.25)–(3.27), we have proved the proposition.  $\square$

## 4 Martin boundary and Martin kernel estimates

In this section we assume that  $D \subset \mathbb{R}^d$  is either a bounded  $C^{1,1}$  domain, or a  $C^{1,1}$  domain with compact complement or a domain above the graph of a bounded  $C^{1,1}$  function. We assume that the  $C^{1,1}$  characteristics of  $D$  is  $(R, \Lambda)$ .

Denote by  $\tilde{Y}^D$  the subordinate killed Brownian motion via a subordinator with Laplace exponent  $\lambda/\phi(\lambda)$ . Let  $\tilde{\mu}(dt)$  be the Lévy measure of the (possibly killed) subordinator with Laplace exponent  $\lambda/\phi(\lambda)$ , the conjugate Bernstein function of  $\phi(\lambda)$ . Since  $\mu((0, \infty)) = \infty$ , we also have  $\tilde{\mu}((0, \infty)) = \infty$ ,

$$\frac{\lambda}{\phi(\lambda)} = u(\infty) + \int_0^\infty (1 - e^{-\lambda t}) \tilde{\mu}(dt)$$

and

$$u(t) = \tilde{\mu}((t, \infty)) + u(\infty). \quad (4.1)$$

(See [33, Corollary 5.5] and the paragraph after it.)

Denote by  $(\tilde{R}_t^D)_{t \geq 0}$  the transition semigroup of  $\tilde{Y}^D$  and by  $\tilde{U}^D$  the potential operator of  $\tilde{Y}^D$ . For any function  $f$  which is excessive for  $W^D$  we define an operator  $\tilde{V}^D$  by

$$\tilde{V}^D f(x) = u(\infty)f(x) + \int_{(0, \infty)} (f(x) - P_t^D f(x)) \tilde{\mu}(dt), \quad x \in D.$$

Let  $G^D(x, y) = \int_0^\infty p^D(t, x, y) dt$  be the Green function of  $W^D$ .

**Lemma 4.1** *For any  $x, y \in D$ , we have*

$$U^D(x, y) = \tilde{V}^D(G^D(\cdot, y))(x).$$

**Proof.** By the semigroup property, for every  $s > 0$ ,

$$\begin{aligned} G^D(x, y) &= \int_0^\infty p^D(t, x, y) dt = \int_0^s p^D(t, x, y) dt + \int_0^\infty p^D(t + s, x, y) dt \\ &= \int_0^s p^D(t, x, y) dt + P_s^D \int_0^\infty p^D(t, \cdot, y)(x) dt = \int_0^s p^D(t, x, y) dt + P_s^D G^D(\cdot, y)(x). \end{aligned}$$

Thus

$$\int_{(0, \infty)} (G^D(x, y) - P_s^D G^D(\cdot, y)(x)) \tilde{\mu}(ds) = \int_{(0, \infty)} \int_0^s p^D(t, x, y) dt \tilde{\mu}(ds). \quad (4.2)$$

Using (4.1) we see that

$$\begin{aligned} \tilde{V}^D(G^D(\cdot, y))(x) &= u(\infty)G^D(x, y) + \int_{(0, \infty)} \int_0^s p^D(t, x, y) dt \tilde{\mu}(ds) \\ &= u(\infty)G^D(x, y) + \int_0^\infty \tilde{\mu}((t, \infty)) p^D(t, x, y) dt \\ &= u(\infty)G^D(x, y) + \int_0^\infty (u(t) - u(\infty)) p^D(t, x, y) dt = U^D(x, y). \end{aligned}$$

□



Note that according to the pointwise version of the Bochner subordination formula one can regard  $-\tilde{V}$  as the generator of  $\tilde{Y}^D$ . This provides an intuitive explanation of Lemma (4.1), namely  $V^D U^D(\cdot, y) = V^D \tilde{V}^D G^D(\cdot, y) = -\Delta G^D(\cdot, y) = -\delta_y$ .

Fix a point  $x_0 \in D$  and define the Martin kernel with respect to  $Y^D$  based at  $x_0$  by

$$M_Y^D(x, y) := \frac{U^D(x, y)}{U^D(x_0, y)}, \quad x, y \in D, \ y \neq x_0. \quad (4.3)$$

We will establish some relation between the Martin kernel for  $Y^D$  and the Martin kernel for  $W^D$ . Define the Martin kernel with respect to  $W^D$  based at  $x_0$  by

$$M^D(x, y) := \frac{G^D(x, y)}{G^D(x_0, y)}, \quad x, y \in D, \ y \neq x_0. \quad (4.4)$$

Since  $D$  is a  $C^{1,1}$  domain, for each  $z \in \partial D$  there exists the limit

$$M^D(x, z) := \lim_{y \rightarrow z} M^D(x, y).$$

In the next lemma, we extend [33, Lemma 5.82] by including our two types of unbounded  $C^{1,1}$  domains and the case  $d = 2$  for bounded  $C^{1,1}$  domains.

**Lemma 4.2** *If  $(y_j)_{j \geq 1}$  is a sequence of points in  $D$  such that  $\lim_{j \rightarrow \infty} y_j = z \in \partial D$ , then for each  $t > 0$  and each  $x \in D$ ,*

$$\lim_{j \rightarrow \infty} P_t^D \left( \frac{G^D(\cdot, y_j)}{G^D(x_0, y_j)} \right) (x) = P_t^D(M^D(\cdot, z))(x).$$

**Proof.** Recall that the  $C^{1,1}$  characteristics of  $D$  is  $(R, \Lambda)$ . Fix  $x \in D$  and let  $R_1 := (R \wedge |x_0 - z| \wedge |x - z|)/4$ . We assume all  $y_j$  are in  $B(z, R_1/2) \cap D$ . For any  $r \in (0, R_1]$ , there exists a ball  $B(A_r(z), r/2) \subset D \cap B(z, r)$ . It is well known (see [1, page 140] and [17, Theorem 7.1]) that there exist  $c_1, \beta > 0$  such that for any  $r \in (0, R_1]$  and any  $(y, w) \in D \cap B(z, r) \times (D \setminus B(z, 2r))$ ,

$$|M^D(w, y) - M^D(w, z)| \leq c_1 M^D(w, A_r(z)) \left( \frac{|y - z|}{r} \right)^\beta. \quad (4.5)$$

Let  $g(w) = |w|^{-d+2}$  be the Newtonian kernel when  $d \geq 3$  and be the logarithmic kernel  $g(x) = (\log \frac{1}{|x|}) \vee 1$  when  $d = 2$ . Using the estimate of  $p^D(t, x, y)$  in (3.1) and the Green function estimates of Brownian motion, we have the following estimates: for every  $t > 0$  there exists a constant  $c_2 = c_2(t, \delta(x), R_1) > 0$  such that

$$p^D(t, x, y) M^D(y, z) \leq c_2 g(y - z) \quad \forall y \in B(z, R_1) \cap D, \quad (4.6)$$

$$p^D(t, x, y) M^D(y, y_j) \leq c_2 g(y - y_j) \quad \forall y \in B(y_j, R_1) \cap D. \quad (4.7)$$

In fact, since

$$\left( \frac{\delta(y)}{|y - y_j|} \wedge \frac{\delta(y)}{\delta(y_j)} \right) \leq 2,$$

for  $d \geq 3$ ,

$$p^D(t, x, y) M^D(y, y_j) \leq c_3(t) \delta(x) \delta(y) \frac{G^D(y, y_j)}{\delta(y_j)}$$

$$\begin{aligned}
&\leq c_4(t, \delta(x)) \frac{\delta(y)}{\delta(y_j)} \left( \frac{\delta(y)}{|y - y_j|} \wedge 1 \right) \left( \frac{\delta(y_j)}{|y - y_j|} \wedge 1 \right) |y - y_j|^{-d+2} \\
&\leq c_4(t, \delta(x)) \frac{\delta(y)}{\delta(y_j)} \left( \frac{\delta(y_j)}{|y - y_j|} \wedge 1 \right) |y - y_j|^{-d+2} \\
&\leq c_4(t, \delta(x)) \left( \frac{\delta(y)}{|y - y_j|} \wedge \frac{\delta(y)}{\delta(y_j)} \right) |y - y_j|^{-d+2} \leq 2c_4(t, \delta(x)) |y - y_j|^{-d+2}.
\end{aligned}$$

This proves (4.7) for  $d \geq 3$ , and by letting  $y_j \rightarrow z$ , we get (4.6) for  $d \geq 3$ . The proofs of (4.6) and (4.7) for  $d = 2$  are similar.

The inequalities (4.6) and (4.7) imply that for every  $r \leq R_1$  and sufficiently large  $j$ ,

$$\int_{D \cap B(z, r)} p^D(t, x, y) (M^D(y, y_j) + M^D(y, z)) dy \leq 2c_2 \int_{B(0, 2r)} g(y) dy. \quad (4.8)$$

Given  $\varepsilon > 0$ , choose  $0 < r_1 \leq R_1$  small such that  $\int_{B(0, 2r_1)} g(y) dy < \varepsilon/(4c_2)$ . For  $y \in D \setminus B(z, r_1)$ , by (4.5) we get that

$$|M^D(y, y_j) - M^D(y, z)| \leq c_2 M^D(y, A_{r_1}(z)) \left( \frac{|y_j - z|}{r_1} \right)^\beta. \quad (4.9)$$

Therefore, using the fact that  $y \rightarrow M^D(y, z)$  is excessive for  $W^D$ , for every large  $j$

$$\begin{aligned}
&|P_t^D \left( \frac{G^D(\cdot, y_j)}{G^D(x_0, y_j)} \right) (x) - P_t^D(M^D(\cdot, z))(x)| \\
&\leq \int_{D \cap B(z, r_1)} p^D(t, x, y) (M^D(y, y_j) + M^D(y, z)) dy + \int_{D \setminus B(z, r_1)} p^D(t, x, y) |M^D(y, y_j) - M^D(y, z)| dy \\
&\leq \varepsilon/2 + c_2 \left( \frac{|y_j - z|}{r_1} \right)^\beta \int_D P_t^D M^D(\cdot, A_{r_1}(z))(y) dy \leq \varepsilon/2 + c_2 \left( \frac{|y_j - z|}{r_1} \right)^\beta M^D(x, A_{r_1}(z)) \leq \varepsilon.
\end{aligned}$$

□

Using the previous lemma, the proof of the next lemma is the same as that of [33, Theorem 5.83(b)]. So we omit the proof.

**Lemma 4.3** *If  $(y_j)_{j \geq 1}$  is a sequence of points in  $D$  converging to  $z \in \partial D$ , then for every  $x \in D$ ,*

$$\lim_{j \rightarrow \infty} \tilde{V}^D \left( \frac{G^D(\cdot, y_j)}{G^D(x_0, y_j)} \right) (x) = \lim_{j \rightarrow \infty} \frac{\tilde{V}^D(G^D(\cdot, y_j))(x)}{G^D(x_0, y_j)} = \tilde{V}^D(M^D(\cdot, z))(x).$$

Let us define the function  $H_Y^D(x, z) := \tilde{V}^D(M^D(\cdot, z))(x)$  on  $D \times \partial D$ . Let  $(y_j)$  be a sequence of points in  $D$  converging to  $z \in \partial D$ , then from Lemma 4.3 we get that

$$H_Y^D(x, z) = \lim_{j \rightarrow \infty} \frac{\tilde{V}^D(G^D(\cdot, y_j))(x)}{G^D(x_0, y_j)} = \lim_{j \rightarrow \infty} \frac{U^D(x, y_j)}{G^D(x_0, y_j)}, \quad (4.10)$$

where the last equality follows from Lemma 4.1. In particular, there exists the limit

$$\lim_{j \rightarrow \infty} \frac{U^D(x_0, y_j)}{G^D(x_0, y_j)} = H_Y^D(x_0, z). \quad (4.11)$$

Now we define a function  $\overline{M}_Y^D$  on  $D \times \partial D$  by

$$\overline{M}_Y^D(x, z) := \frac{H_Y^D(x, z)}{H_Y^D(x_0, z)}, \quad x \in D, z \in \partial D. \quad (4.12)$$

From the definition above and (4.10)–(4.11), we can easily see that

$$\lim_{D \ni y \rightarrow z} \frac{U^D(x, y)}{U^D(x_0, y)} = \overline{M}_Y^D(x, z), \quad x \in D, z \in \partial D. \quad (4.13)$$

Thus we have proved the following result.

**Proposition 4.4** *The function  $M_Y^D(\cdot, \cdot)$  can be extended from  $D \times D$  to  $D \times \overline{D}$  so that for each  $z \in \partial D$  we have that*

$$\overline{M}_Y^D(x, z) = \lim_{y \rightarrow z} M_Y^D(x, y) = \lim_{y \rightarrow z} \frac{U^D(x, y)}{U^D(x_0, y)}.$$

The following two types of sharp two-sided estimates for  $\overline{M}_Y^D(x, z)$  now follow easily from Theorems 3.1 and 3.2.

**Theorem 4.5** *Assume that  $\phi$  is a Bernstein function satisfying (A1)–(A5). Let  $D \subset \mathbb{R}^d$  be a bounded  $C^{1,1}$  domain, or a  $C^{1,1}$  domain with compact complement or domain above the graph of a bounded  $C^{1,1}$  function. For every  $M > 0$  and  $z \in \partial D$ , there exists a constant  $c = c(M, z) \geq 1$  such that for all  $x \in D$  with  $|x - z| \leq M$ ,*

$$c^{-1} \frac{\delta(x)\phi'(|x - z|^{-2})}{|x - z|^{d+4}\phi(|x - z|^{-2})^2} \leq \overline{M}_Y^D(x, z) \leq c \frac{\delta(x)\phi'(|x - z|^{-2})}{|x - z|^{d+4}\phi(|x - z|^{-2})^2}. \quad (4.14)$$

Note that the constant  $c$  in Theorem 4.5 will in general depend on  $z \in \partial D$ . This is inconsequential, because the point  $z$  will always be fixed.

**Theorem 4.6** *Assume that  $\phi$  is a Bernstein function satisfying (A1)–(A3) and (A6). (1) Let  $D \subset \mathbb{R}^d$  be a domain above the graph of a bounded  $C^{1,1}$  function. There exists a constant  $c_1 \geq 1$  such that for all  $x \in D$  and  $z \in \partial D$ ,*

$$c_1^{-1} \delta(x) \frac{u(|x - z|^2)|x_0 - z|^d}{u(|x_0 - z|^2)|x - z|^d} \leq \overline{M}_Y^D(x, z) \leq c_1 \delta(x) \frac{u(|x - z|^2)|x_0 - z|^d}{u(|x_0 - z|^2)|x - z|^d}. \quad (4.15)$$

(2) Let  $D \subset \mathbb{R}^d$  be a  $C^{1,1}$  domain with compact complement. There exists a constant  $c_2 \geq 1$  such that for all  $x \in D$  and  $z \in \partial D$ ,

$$\begin{aligned} c_2^{-1} \left( \frac{\delta(x)}{|x - z| \wedge 1} \wedge 1 \right) \left( \frac{|x_0 - z| \wedge 1}{|x - z| \wedge 1} \right) \frac{u(|x - z|^2)|x_0 - z|^{d-2}}{u(|x_0 - z|^2)|x - z|^{d-2}} &\leq \overline{M}_Y^D(x, z) \\ &\leq c_2 \left( \frac{\delta(x)}{|x - z| \wedge 1} \wedge 1 \right) \left( \frac{|x_0 - z| \wedge 1}{|x - z| \wedge 1} \right) \frac{u(|x - z|^2)|x_0 - z|^{d-2}}{u(|x_0 - z|^2)|x - z|^{d-2}}. \end{aligned} \quad (4.16)$$

**Remark 4.7** (1) Theorem 4.5 in particular implies that  $\overline{M}_Y^D(\cdot, z_1)$  differs from  $\overline{M}_Y^D(\cdot, z_2)$  if  $z_1$  and  $z_2$  are two different points on  $\partial D$ .

(2) From Theorem 4.6, we have  $\lim_{D \ni x \rightarrow \infty} \overline{M}_Y^D(x, z) = 0$  for any  $z \in \partial D$ . In fact, for  $|x - z| \geq |z - x_0|$  we have  $u(|x - z|) \leq u(|x_0 - z|)$ . It is clear that

$$\limsup_{D \ni x \rightarrow \infty} \left( \delta(x) \frac{|x_0 - z|^2}{|x - z|^2} + \frac{|x_0 - z| \wedge 1}{|x - z| \wedge 1} \right) \leq \limsup_{D \ni x \rightarrow \infty} \left( \frac{|x_0 - z|^2}{|x - z|} + \frac{|x_0 - z| \wedge 1}{|x - z| \wedge 1} \right) < \infty.$$

Thus, in both cases,

$$\limsup_{D \ni x \rightarrow \infty} \overline{M}_Y^D(x, z) \leq c \limsup_{D \ni x \rightarrow \infty} \frac{u(|x - z|^2)|x_0 - z|^{d-2}}{u(|x_0 - z|^2)|x - z|^{d-2}} \leq c \limsup_{D \ni x \rightarrow \infty} \frac{|x_0 - z|^{d-2}}{|x - z|^{d-2}} = 0. \quad (4.17)$$

Using the continuity of  $U^D$  in the extended sense (Proposition 3.3) and the upper bound in (2.16), one can check that  $Y^D$  satisfies Hypothesis (B) in [24]. Therefore,  $D$  has a Martin boundary  $\partial_M D$  with respect to  $Y^D$  satisfying the following properties:

- (M1)  $D \cup \partial_M D$  is a compact metric space (with the metric denoted by  $d$ );
- (M2)  $D$  is open and dense in  $D \cup \partial_M D$ , and its relative topology coincides with its original topology;
- (M3)  $M_Y^D(x, \cdot)$  can be uniquely extended to  $\partial_M D$  in such a way that
  - (a)  $M_Y^D(x, y)$  converges to  $M_Y^D(x, w)$  as  $y \rightarrow w \in \partial_M D$  in the Martin topology;
  - (b) for each  $w \in D \cup \partial_M D$  the function  $x \rightarrow M_Y^D(x, w)$  is excessive with respect to  $Y^D$ ;
  - (c) the function  $(x, w) \rightarrow M_Y^D(x, w)$  is jointly continuous on  $D \times ((D \setminus \{x_0\}) \cup \partial_M D)$  in the Martin topology and
  - (d)  $M_Y^D(\cdot, w_1) \neq M_Y^D(\cdot, w_2)$  if  $w_1 \neq w_2$  and  $w_1, w_2 \in \partial_M D$ .

Recall that a positive harmonic function  $f$  for  $Y^D$  is minimal if, whenever  $h$  is a positive harmonic function for  $Y^D$  with  $h \leq f$  on  $D$ , one must have  $f = ch$  for some constant  $c$ . A point  $z \in \partial_M D$  is called a minimal Martin boundary point if  $M_Y^D(\cdot, z)$  is a minimal harmonic function for  $Y^D$ . The minimal Martin boundary of  $Y^D$  is denoted by  $\partial_m D$ .

We will say that a point  $w \in \partial_M D$  is a finite Martin boundary point if there exists a bounded sequence  $(y_n)_{n \geq 1} \subset D$  converging to  $w$  in the Martin topology. Recall that a point  $w$  on the Martin boundary  $\partial_M D$  of  $D$  is said to be associated with  $z \in \partial D$  if there is a sequence  $(y_n)_{n \geq 1} \subset D$  converging to  $w$  in the Martin topology and to  $z$  in the Euclidean topology. The set of Martin boundary points associated with  $z$  is denoted by  $\partial_M^z D$ .

By using Proposition 4.4, the proof of next lemma is same as that of [22, Lemma 3.6]. Thus we omit it.

**Proposition 4.8** *For any  $z \in \partial D$ ,  $\partial_M^z D$  consists of exactly one point  $w$  and  $M_Y^D(\cdot, w) = \overline{M}_Y^D(\cdot, z)$ .*

Because of the proposition above, we will also use  $z$  to denote the point on the Martin boundary  $\partial_M^z D$  associated with  $z \in \partial D$ . Note that it follows from the proof of [22, Lemmas 3.6] that if  $(y_n)_{n \geq 1}$  converges to  $z \in \partial D$  in the Euclidean topology, then it also converges to  $z$  in the Martin topology.

In the remainder of this section, we fix  $z \in \partial D$ . The proof of the next result is same as that of [22, Lemma 3.8]. Thus we omit the proof.

**Lemma 4.9** *For every bounded open  $O \subset \overline{O} \subset D$  and every  $x \in D$ ,  $M_Y^D(Y_{\tau_O}^D, z)$  is  $\mathbb{P}_x$ -integrable.*

Using the results above, we can get the following result.

**Lemma 4.10** *Suppose that  $\phi$  is a Bernstein function satisfying (A1)–(A6). For any  $x \in D$  and  $r \in (0, R \wedge (\delta(x)/2)]$ ,*

$$M_Y^D(x, z) = \mathbb{E}_x[M_Y^D(Y_{\tau_{B(x,r)}}^D, z)].$$

**Proof.** Recall that  $D$  satisfies the interior and exterior balls conditions with radius  $R$ . Thus, for all  $r \in (0, R]$ , there is a ball  $B(A_r(z), r/2) \subset D \cap B(z, r)$ . Fix  $x \in D$  and a positive  $r < R \wedge \frac{\delta(x)}{2}$ . Let

$$\eta_m := 2^{-2m}r \quad \text{and} \quad z_m = A_{\eta_m}(z), \quad m = 0, 1, \dots$$

Note that

$$B(z_m, \eta_{m+1}) \subset D \cap B(z, 2^{-1}\eta_m) \subset D \cap B(z, \eta_m) \subset D \cap B(z, r) \subset D \setminus B(x, r)$$

for all  $m \geq 0$ . Thus by the harmonicity of  $M_Y^D(\cdot, z_m)$ , we have

$$M_Y^D(x, z_m) = \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m) \right].$$

Choose  $m_0 = m_0 \geq 2$  such that  $\eta_{m_0} < \delta(x_0)/4$ .

To prove the lemma, it suffices to show that  $\{M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m) : m \geq m_0\}$  is  $\mathbb{P}_x$ -uniformly integrable. Fix an arbitrary  $\varepsilon > 0$ . We first note that if  $D$  is unbounded, by Theorem 3.2 there exists  $L \geq 2r \vee 2$  such that for every  $m \geq m_0$  and  $w \in D \setminus B(z, L)$ ,

$$\begin{aligned} \frac{U^D(w, z_m)}{U^D(x_0, z_m)} &\leq \frac{c}{\delta(z_m)} \left( \frac{\delta(z_m)}{|w - z_m| \wedge 1} \wedge 1 \right) \left( \frac{\delta(w)}{|w - z_m| \wedge 1} \wedge 1 \right) \frac{u(|w - z_m|^2)}{|w - z_m|^{d-2}} \\ &\leq \frac{c}{\delta(z_m)} (\delta(z_m) \wedge 1) (\delta(w) \wedge 1) \frac{u(|w - z_m|^2)}{|w - z_m|^{d-2}} \leq c \frac{u(|w - z_m|^2)}{|w - z_m|^{d-2}} \\ &\leq c \frac{\phi'(|w - z_m|^{-2})}{|w - z_m|^{d+2} \phi(|w - z_m|^{-2})^2} \leq c \frac{\phi'((L/2)^{-2})}{(L/2)^{d+2} \phi((L/2)^{-2})^2} \leq \frac{\varepsilon}{4}. \end{aligned}$$

In the above inequalities, we have used Lemma 2.1(b). If  $D$  is a bounded domain we simply take  $L = 2\text{diam}(D)$  so that  $D \setminus B(z, L) = \emptyset$ . Thus

$$\mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); Y_{\tau_{B(x,r)}}^D \in D \setminus B(z, L) \right] \leq \frac{\varepsilon}{4}. \quad (4.18)$$

By Theorem 3.1, there exist  $m_1 \geq m_0$  and  $c_1 = c_1(L) > 0$  such that for every  $w \in (D \cap B(z, L)) \setminus B(z, \eta_m)$  and  $y \in D \cap B(z, \eta_{m+1})$ ,

$$M_Y^D(w, z_m) \leq c_1 M_Y^D(w, y), \quad m \geq m_1.$$

Letting  $y \rightarrow z$  we get

$$M_Y^D(w, z_m) \leq c_1 M_Y^D(w, z), \quad m \geq m_1, w \in (D \cap B(z, L)) \setminus B(z, \eta_m). \quad (4.19)$$

Since  $M_Y^D(Y_{\tau_{B(x,r)}}^D, z)$  is  $\mathbb{P}_x$ -integrable by Lemma 4.9, there is an  $N_0 = N_0(\varepsilon) > 1$  such that

$$\mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z); M_Y^D(Y_{\tau_{B(x,r)}}^D, z) > N_0/c_1 \right] < \frac{\varepsilon}{2c_1}. \quad (4.20)$$

By (4.18), (4.19) and (4.20),

$$\begin{aligned}
& \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m) > N_0 \text{ and } Y_{\tau_{B(x,r)}}^D \in D \setminus B(z, \eta_m) \right] \\
& \leq \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m) > N_0 \text{ and } Y_{\tau_{B(x,r)}}^D \in (D \cap B(z, L)) \setminus B(z, \eta_m) \right] \\
& \quad + \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); Y_{\tau_{B(x,r)}}^D \in D \setminus B(z, L) \right] \\
& \leq c_1 \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z); c_1 M_Y^D(Y_{\tau_{B(x,r)}}^D, z) > N_0 \right] + \frac{\varepsilon}{4} < c_1 \frac{\epsilon}{2c_1} + \frac{\varepsilon}{4} = \frac{3\epsilon}{4}.
\end{aligned}$$

By (3.24), we have for  $m \geq m_1$ ,

$$\begin{aligned}
& \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); Y_{\tau_{B(x,r)}}^D \in D \cap B(z, \eta_m) \right] \\
& = \int_{D \cap B(z, \eta_m)} M_Y^D(w, z_m) K^{D, B(x, r)}(x, w) dw \\
& \leq c_2 \phi(r^{-2})^{-1} \int_{D \cap B(z, \eta_m)} M_Y^D(w, z_m) \delta(w) \frac{\phi'(|w-x|-r)^{-2}}{(|w-x|-r)^{d+3}} dw.
\end{aligned}$$

Since  $|w-x| \geq |x-z| - |z-w| \geq \delta(x) - \eta_m \geq \frac{7}{4}r$ , applying Lemma 2.1(a)–(c), we get that

$$\begin{aligned}
& \mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); Y_{\tau_{B(x,r)}}^D \in D \cap B(z, \eta_m) \right] \\
& \leq c_3 r^{-d-3} \phi'((3r/4)^{-2}) \phi((3r/4)^{-2})^{-1} \int_{D \cap B(z, \eta_m)} M_Y^D(w, z_m) \delta(w) dw \\
& \leq c_4 r^{-d-3} \phi'(r^{-2}) \phi(r^{-2})^{-1} U^D(x_0, z_m)^{-1} \int_{D \cap B(z, \eta_m)} U^D(w, z_m) \delta(w) dw. \tag{4.21}
\end{aligned}$$

Note that, by Theorem 3.1 ,

$$U^D(x_0, z_m)^{-1} \leq \frac{c_5}{\eta_m} \tag{4.22}$$

and by (2.16)

$$\begin{aligned}
& \int_{D \cap B(z, \eta_m)} \delta(w) U^D(w, z_m) dw \leq \int_{D \cap B(z, \eta_m)} \delta(w) G_X(w, z_m) dw \\
& \leq c_6 \eta_m \int_{D \cap B(z, \eta_m)} \frac{\phi'(|w-z_m|^{-2})}{|w-z_m|^{d+2} \phi(|w-z_m|^{-2})^2} dw \\
& \leq c_6 \eta_m \int_{B(z_m, 2\eta_m)} \frac{\phi'(|w-z_m|^{-2})}{|w-z_m|^{d+2} \phi(|w-z_m|^{-2})^2} dw \\
& = c_6 \eta_m \int_{B(0, 2\eta_m)} \frac{\phi'(|w|^{-2})}{|w|^{d+2} \phi(|w|^{-2})^2} dw = c_7 \eta_m \int_0^{2\eta_m} \frac{\phi'(r^{-2})}{r^3 \phi(r^{-2})^2} dr \\
& = c_7 \eta_m \int_0^{2\eta_m} \frac{d}{dr} \left( \frac{1}{\phi(r^{-2})} \right) dr \leq c_8 \eta_m \phi((2\eta_m)^{-2})^{-1}. \tag{4.23}
\end{aligned}$$

It follows from (4.21)–(4.23) that

$$\mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); Y_{\tau_{B(x,r)}}^D \in D \cap B(z, \eta_m) \right]$$

$$\leq c_9 r^{-d-3} \phi'(r^{-2}) \phi(r^{-2})^{-1} \frac{1}{\phi((2\eta_m)^{-2})} \leq \frac{c(r)}{\phi((2\eta_m)^{-2})}.$$

Thus there exists  $m_2 \geq m_1$  such that for all  $m \geq m_2$ ,

$$\mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); Y_{\tau_{B(x,r)}}^D \in D \cap B(z, \eta_m) \right] \leq \frac{\epsilon}{4}.$$

Consequently, for all  $m \geq m_2$ ,

$$\mathbb{E}_x \left[ M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m); M^D(Y_{\tau_{B(x,r)}}^D, z_m) > N \right] \leq \epsilon,$$

which implies that  $\{M_Y^D(Y_{\tau_{B(x,r)}}^D, z_m) : m \geq m_0\}$  is  $\mathbb{P}_x$ -uniformly integrable.  $\square$

Using this, we can easily get the following

**Theorem 4.11** *Suppose that  $\phi$  is a Bernstein function satisfying (A1)–(A6). The function  $M^D(\cdot, z)$  is harmonic in  $D$  with respect to  $Y^D$ .*

**Proof.** The proof is the same as that of [22, Theorem 3.10].  $\square$

**Theorem 4.12** *Suppose that  $\phi$  is a Bernstein function satisfying (A1)–(A6). Every point  $z$  on  $\partial D$  is a minimal Martin boundary point.*

**Proof.** Fix  $z \in \partial D$  and let  $h$  be a positive harmonic function for  $Y^D$  such that  $h \leq M_Y^D(\cdot, z)$ . By the Martin representation in [24], there is a finite measure on  $\partial_M D$  such that

$$h(x) = \int_{\partial_M D} M_Y^D(x, w) \mu(dw) = \int_{\partial_M D \setminus \{z\}} M_Y^D(x, w) \mu(dw) + M_Y^D(x, z) \mu(\{z\}).$$

In particular,  $\mu(\partial_M D) = h(x_0) \leq M_Y^D(x_0, z) = 1$  (because of the normalization at  $x_0$ ). Hence,  $\mu$  is a sub-probability measure.

For  $\epsilon > 0$ , put  $K_\epsilon := \{w \in \partial_M D : d(w, z) \geq \epsilon\}$ . Then  $K_\epsilon$  is a compact subset of  $\partial_M D$ . Define

$$u(x) := \int_{K_\epsilon} M_Y^D(x, w) \mu(dw). \quad (4.24)$$

Then  $u$  is a positive harmonic function with respect to  $Y^D$  satisfying

$$u(x) \leq h(x) - \mu(\{z\}) M_Y^D(x, z) \leq (1 - \mu(\{z\})) M_Y^D(x, z). \quad (4.25)$$

By (M3)(c), our estimates in Theorems 4.5 and 4.6 and the fact  $\lim_{D \ni x \rightarrow \infty} M_Y^D(x, z) = 0$  (cf. Remark 4.7) we see from (4.24) and (4.25) that  $u$  is bounded,  $\lim_{D \ni x \rightarrow w} u(x) = 0$  for every  $w \in \partial D$  and  $\lim_{D \ni x \rightarrow \infty} u(x) = 0$ . Therefore by the harmonicity of  $u$ ,  $u \equiv 0$  in  $D$ .

We see from (4.24) that  $\nu = \mu|_{K_\epsilon} = 0$ . Since  $\epsilon > 0$  was arbitrary and  $\partial_M D \setminus \{z\} = \cup_{\epsilon > 0} K_\epsilon$ , we see that  $\mu|_{\partial_M D \setminus \{z\}} = 0$ . Hence  $h = \mu(\{z\}) M_Y^D(\cdot, z)$  showing that  $M_Y^D(\cdot, z)$  is minimal.  $\square$

Combining Remark 4.7(1) and Theorem 4.12, we conclude that

**Theorem 4.13** *Suppose that  $\phi$  is a Bernstein function satisfying (A1)–(A6). The finite part of the minimal Martin boundary of  $D$  and the finite part of the Martin boundary of  $D$  both coincide with the Euclidean boundary  $\partial D$  of  $D$ .*

We conclude this section with following inequality, which will be used in Section 6.

**Corollary 4.14** *Fix  $z \in \partial D$  and assume that  $x_0 \in D \cap B(z, R)$  satisfies  $R/4 < \delta(x_0) < R$  and  $M_Y^D$  is the Martin kernel of  $D$  based on  $x_0$ . Then there exists  $c = c(z) > 0$  such that for all  $x, y \in B(z, R/4)$  with  $\frac{3}{4}|x - z| \leq |x - y|$ ,*

$$\frac{U^D(x, y)}{M_Y^D(x, z)} \leq c U^D(x_0, y). \quad (4.26)$$

**Proof.** It follows from Theorem 3.1 and Theorem 4.5 that

$$\begin{aligned} U^D(x, y) &\asymp \delta(x)\delta(y)|x - y|^{-d-4}\phi'(|x - y|^{-2})\phi(|x - y|^{-2})^{-2}, \\ M_Y^D(x, z) &\asymp \delta(x)|x - z|^{-d-4}\phi'(|x - z|^{-2})\phi(|x - z|^{-2})^{-2}, \\ U^D(x_0, y) &\asymp \delta(y)|x_0 - y|^{-d-4}\phi'(|x_0 - y|^{-2})\phi(|x_0 - y|^{-2})^{-2} \asymp \delta(y). \end{aligned}$$

Since  $|x_0 - y| \geq R/4$  and  $r \mapsto r^{-d-4}\phi'(r^{-2})\phi(r^{-2})^{-2}$  is decreasing, we can estimate  $U^D(x_0, y) \geq c_1\delta(y)$ . Using the monotonicity of  $r \mapsto r^{-d-4}\phi'(r^{-2})\phi(r^{-2})^{-2}$ , we get

$$\frac{\phi'(|x - y|^{-2})}{|x - y|^{d+4}\phi(|x - y|^{-2})^2} \leq c \frac{\phi'((3|x - z|)/4)^{-2}}{((3|x - z|)/4)^{d+4}\phi((3|x - z|)/4)^{-2}}.$$

Applying Lemma 2.1(c) we get that  $U^D(x, y)/M_Y^D(x, z) \leq c_1\delta(y)$ . This completes the proof.  $\square$

## 5 Quasi-additivity of capacity

Throughout this section we assume that  $\phi$  is a Bernstein function satisfying **(A1)**–**(A5)**. Let  $\text{Cap}$  denote the capacity with respect to the subordinate Brownian motion  $X$  and  $\text{Cap}_D$  the capacity with respect to the subordinate killed Brownian motion  $Y^D$ . The goal of this section is to prove that  $\text{Cap}_D$  is quasi-additive with respect to Whitney decompositions of  $D$ .

We start with the following inequality: There exist positive constants  $c_1 < c_2$  such that

$$c_1 r^d \phi(r^{-2}) \leq \text{Cap}(\overline{B(0, r)}) \leq c_2 r^d \phi(r^{-2}), \quad \text{for every } r \in (0, 1]. \quad (5.1)$$

Using (2.16), the proof of (5.1) is the same as that of [23, Proposition 5.2]. Thus we omit the proof.

For any open set  $D \subset \mathbb{R}^d$ , let  $\mathcal{S}(D)$  denote the collection of all excessive functions with respect to  $Y^D$  and let  $\mathcal{S}^c(D)$  be the family of positive functions in  $\mathcal{S}(D)$  which are continuous in the extended sense. For any  $v \in \mathcal{S}(D)$  and  $E \subset D$ , the reduced function of  $v$  relative to  $E$  in  $D$  is defined by

$$R_v^E(x) = \inf\{w(x) : w \in \mathcal{S}(D) \text{ and } w \geq v \text{ on } E\}, \quad x \in \mathbb{R}^d. \quad (5.2)$$

The lower semi-continuous regularization  $\widehat{R}_v^E$  of  $R_v^E$  is called the balayage of  $v$  relative to  $E$  in  $D$ . Note that the killed Brownian motion  $W^D$  is a strongly Feller process. Thus it follows by [5, Proposition V.3.3] that the semigroup of  $Y^D$  also has strong Feller property. So it follows easily from [5, Proposition V.2.2] that the cone of excessive functions  $\mathcal{S}(D)$  is a balayage space in the sense of [5].

In the remainder of this section we assume that  $D \subset \mathbb{R}^d$  is either a bounded  $C^{1,1}$  domain, or a  $C^{1,1}$  domain with compact complement or a domain above the graph of a bounded  $C^{1,1}$  function.



Given  $v \in \mathcal{S}^c(D)$ , define a kernel  $k_v : D \times D \rightarrow [0, \infty]$  by

$$k_v(x, y) := \frac{U^D(x, y)}{v(x)v(y)}, \quad x, y \in D. \quad (5.3)$$

We will later consider  $v(y) = U^D(y, x_0) \wedge 1$ . Note that  $k_v(x, y)$  is jointly lower semi-continuous on  $D \times D$  by the joint lower semi-continuity of  $U^D$ , cf. Proposition 3.3, and the assumptions that  $v$  is positive and continuous in the extended sense. For a measure  $\lambda$  on  $D$  let  $\lambda_v(dy) := \lambda(dy)/v(y)$ . Then

$$k_v \lambda(x) := \int_D k_v(x, y) \lambda(dy) = \int_D \frac{U^D(x, y)}{v(x)v(y)} \lambda(dy) = \frac{1}{v(x)} \int_D U^D(x, y) \frac{\lambda(dy)}{v(y)} = \frac{1}{v(x)} U^D \lambda_v(dy).$$

We define a capacity with respect to the kernel  $k_v$  as follows:

$$\mathcal{C}_v(E) := \inf\{\|\lambda\| : k_v \lambda \geq 1 \text{ on } E\}, \quad E \subset D,$$

where  $\|\lambda\|$  denotes the total mass of the measure  $\lambda$  on  $D$ . The following dual representation of the capacity of compact sets can be found in [15, Théorème 1.1]:

$$\mathcal{C}_v(K) = \sup\{\mu(K) : \mu(D \setminus K) = 0, k_v \mu \leq 1 \text{ on } D\}. \quad (5.4)$$

For a compact set  $K \subset D$ , consider the balayage  $\widehat{R}_v^K$ . Being a potential,  $\widehat{R}_v^K = U^D \lambda^{K,v}$  for a measure  $\lambda^{K,v}$  supported in  $K$ . Recall that  $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$  is the Dirichlet form associated with  $Y^D$ . Define the Green energy of  $K$  (with respect to  $v$ ) by

$$\gamma_v(K) := \int_D \int_D U^D(x, y) \lambda^{K,v}(dx) \lambda^{K,v}(dy) = \int_D U^D \lambda^{K,v}(x) \lambda^{K,v}(dx) = \mathcal{E}^D(U^D \lambda^{K,v}, U^D \lambda^{K,v}).$$

As usual, this definition of energy is extended first to open and then to Borel subsets of  $D$ . By following the proof of [23, Proposition 5.3] we see that for all Borel subsets  $E \subset D$  it holds that

$$\gamma_v(E) = \mathcal{C}_v(E). \quad (5.5)$$

Note that in case  $v \equiv 1$ ,  $\gamma_1(E) = \mathcal{C}_1(E) = \text{Cap}_D(E)$ .

Let  $\{Q_j\}_{j \geq 1}$  be a Whitney decomposition of  $D$ . Recall that  $x_j$  is the center of  $Q_j$  and  $Q_j^*$  the interior of the double of  $Q_j$ . Then  $\{Q_j, Q_j^*\}$  is a quasi-disjoint decomposition of  $D$  in the sense of [3, pp. 146-147].

**Definition 5.1** A kernel  $k : D \times D \rightarrow [0, +\infty]$  is said to satisfy the local Harnack property with localization constant  $r_1 > 0$  with respect to  $\{Q_j, Q_j^*\}$  if

$$k(x, y) \asymp k(x', y), \text{ for all } x, x' \in Q_j \text{ and all } y \in D \setminus Q_j^*, \quad (5.6)$$

for all cubes  $Q_j$  of diameter less than  $r_1$ .

**Definition 5.2** A function  $v : D \rightarrow (0, \infty)$  is said to satisfy the local scale invariant Harnack inequality with localization constant  $r_1 > 0$  with respect to  $\{Q_j\}$  if there exists  $c > 0$  such that

$$\sup_{Q_j} v \leq c \inf_{Q_j} v, \quad \text{for all } Q_j \text{ with } \text{diam}(Q_j) < r_1. \quad (5.7)$$

**Lemma 5.3** *If  $v \in \mathcal{S}^c(D)$  satisfies the local scale invariant Harnack inequality with localization constant  $r_1 > 0$  with respect to  $\{Q_j\}$ , then the kernel  $k_v$  satisfies the local Harnack property with localization constant  $r_1 > 0$  with respect to  $\{Q_j, Q_j^*\}$ .*

**Proof.** This is an immediate consequence of Corollary 3.4(i).  $\square$

Typical examples of positive continuous excessive functions  $v$  that satisfy the scale invariant Harnack inequality are functions  $v \equiv 1$  and  $v = U^D(\cdot, x_0) \wedge c$  with  $x_0 \in D$  and  $c > 0$  fixed.

**Lemma 5.4** *For every  $M > 0$ , there exists a constant  $c = c(M) \in (0, 1)$  such that*

$$c \text{Cap}_D(Q_j) \leq \text{Cap}(Q_j) \leq \text{Cap}_D(Q_j) \quad (5.8)$$

*for all Whitney cubes whose diameter is less than  $M$ .*

**Proof.** By (5.4) and (5.5) we have that for every compact set  $K \subset D$ ,

$$\text{Cap}_D(K) = \sup\{\mu(K) : \text{supp}(\mu) \subset K, U^D\mu \leq 1 \text{ on } D\}.$$

If  $\text{supp}(\mu) \subset K$  and  $G_X\mu \leq 1$  on  $\mathbb{R}^d$ , then clearly  $U^D\mu \leq 1$  on  $D$ . This implies that  $\text{Cap}(K) \leq \text{Cap}_D(K)$  for all compact subset  $K \subset D$ , in particular for each Whitney cube  $Q_j$ .

Let  $\mu$  be the capacitary measure of  $Q_j$  (with respect to  $Y^D$ ), i.e.,  $\mu(Q_j) = \text{Cap}_D(Q_j)$  and  $U^D\mu \leq 1$ . Then by Corollary 3.4(ii) for every  $x \in Q_j$  we have

$$1 \geq U^D\mu(x) = \int_{Q_j} U^D(x, y) \mu(dy) \geq \int_{Q_j} c G_X(x, y) \mu(dy) = G_X(c\mu)(x).$$

By the maximum principle it follows that  $G_X(c\mu) \leq 1$  everywhere on  $\mathbb{R}^d$ . Hence,  $\text{Cap}(Q_j) \geq (c\mu)(Q_j) = c \text{Cap}_D(Q_j)$ .  $\square$

**Lemma 5.5** *Suppose that  $v \in \mathcal{S}^c(D)$  is a function satisfying the local scale invariant Harnack inequality with localization constant  $r_1 > 0$  with respect to  $Y^D$ . Then for every  $Q_j$  of diameter less than  $r_1$  and every  $E \subset Q_j$  it holds that*

$$\gamma_v(E) \asymp v(x_j)^2 \text{Cap}_D(E). \quad (5.9)$$

**Proof.** The proof is same as the proof of [23, Lemma 5.8(i)].  $\square$

**Definition 5.6** *Let  $\{Q_j\}$  be a Whitney decomposition of  $D$  and  $v \in \mathcal{S}^c(D)$ . A Borel measure  $\sigma$  on  $D$  is locally comparable to the capacity  $\mathcal{C}_v$  with respect to  $\{Q_j\}$  at  $z \in \partial D$  if there exists  $r, c > 0$  such that*

$$\begin{aligned} \sigma(Q_j) &\asymp \mathcal{C}_v(Q_j), \quad \text{for all } Q_j \text{ with } Q_j \cap B(z, r) \neq \emptyset, \\ \sigma(E) &\leq c \mathcal{C}_v(E), \quad \text{for all Borel } E \subset D \cap B(z, 2r). \end{aligned}$$

Recall that  $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$  is the Dirichlet form associated with  $Y^D$ .

**Lemma 5.7** (*Local Hardy's inequality*) *There exist constants  $c > 0$  and  $r > 0$  such that for every  $v \in \mathcal{D}(\mathcal{E}^D)$  and  $z \in \partial D$ ,*

$$\mathcal{E}^D(v, v) \geq c \int_{D \cap B(z, r)} v(x)^2 \phi(\delta(x)^{-2}) dx. \quad (5.10)$$

**Proof.** Since  $D$  is a  $C^{1,1}$  domain, there exist  $b_1 > 1$ ,  $R_1 > 0$  and a cone  $C$  whose vertex is at the origin, such that for every  $z \in \partial D$  and  $x \in D \cap B(z, b_1 R_1/2)$ , there exists  $\widehat{C}$ , which is a rotation of  $C$ , such that

$$(\widehat{C} + x) \cap \{b_1 \delta(x) < |x - y| < R_1\} \subset D^c. \quad (5.11)$$

Choose  $r \in (0, b_1 R_1/2)$  small that  $\phi((b_1 r)^{-2}) \geq 2\phi(R_1^{-2})$ .

Fix  $v \in \mathcal{D}(\mathcal{E}^D)$  and  $z \in \partial D$ . By (2.4) and (2.7),

$$\mathcal{E}^D(v, v) \geq \int_{D \cap B(z, r)} v(x)^2 \kappa_D(x) dx \geq \int_{D \cap B(z, r)} v(x)^2 \kappa_D^X(x) dx.$$

Let  $x \in D \cap B(z, r)$ . By (2.6), (5.11), and the lower bound in (2.17),

$$\begin{aligned} \kappa_D^X(x) &= \int_{D^c} j(x - y) dy \geq \int_{(\widehat{C} + x) \cap \{b_1 \delta(x) < |x - y| < R_1\}} j(x - y) dy \\ &\geq c_1 \int_{(\widehat{C} + x) \cap \{b_1 \delta(x) < |x - y| < R_1\}} |x - y|^{-d-2} \phi'(|x - y|^{-2}) dy \\ &\geq c_2 \int_{b_1 \delta(x)}^{R_1} -\frac{d}{ds} (\phi(s^{-2})) ds = c_2 (\phi((b_1 \delta(x))^{-2}) - \phi(R_1^{-2})) \\ &= 2^{-1} c_2 \phi((b_1 \delta(x))^{-2}) \geq c_3 \phi(\delta(x)^{-2}). \end{aligned}$$

In the second to last inequality we used  $\phi((b_1 \delta(x))^{-2}) \geq \phi((b_1 r)^{-2}) \geq 2\phi(R_1^{-2})$  and, in the last inequality we used (2.1).  $\square$

For  $v \in \mathcal{S}^c(D)$ , define

$$\sigma_v(E) := \int_E v(x)^2 \phi(\delta(x)^{-2}) dx, \quad E \subset D.$$

**Proposition 5.8** *Let  $v \in \mathcal{S}^c(D)$  satisfy the local scale invariant Harnack inequality with localization constant  $r_1 > 0$  with respect to the Whitney decomposition  $\{Q_j\}$ . Then  $\sigma_v$  is locally comparable to the capacity  $\mathcal{C}_v$  with respect to  $\{Q_j\}$  for every  $z \in D$ .*

**Proof.** Fix  $z \in \partial D$  and let  $\tilde{r} = (r_1 \wedge r_2)/2$  where  $r_2$  is the constant  $r$  in Lemma 5.7. Since  $v$  satisfies the local scale invariant Harnack inequality with localization constant  $r_1$ , we have  $v \asymp v(x_j)$  on any  $Q_j$  of diameter less than  $r_1$ . By Lemma 5.5,  $\gamma_v(Q_j) \asymp v(x_j)^2 \text{Cap}_D(Q_j)$  for any  $Q_j$  of diameter less than  $\tilde{r}$ . On the other hand, by Lemma 5.4 and (5.1),

$$\sigma_v(Q_j) = \int_{Q_j} v(x)^2 \phi(\delta(x)^{-2}) dx \asymp v(x_j)^2 \phi((\text{diam}(Q_j))^{-2} |Q_j|) \asymp \text{Cap}(D) \asymp \text{Cap}_D(Q_j)$$

for all  $Q_j$  with  $Q_j \cap B(z, \tilde{r}) \neq \emptyset$ . Thus  $\gamma_v(Q_j) \asymp \text{Cap}_D(Q_j)$ .

Using local Hardy's inequality, Lemma 5.7, for any Borel subset  $E \subset D$  and compact  $K \subset E \cap B(z, 2\tilde{r})$ ,

$$\begin{aligned}\gamma_v(E) &\geq \gamma_v(K) = \mathcal{E}^D(U^D \lambda^{K,v}, U^D \lambda^{K,v}) \geq c_1 \int_K (U^D \lambda^{K,v})(x)^2 \phi(\delta(x)^{-2}) dx \\ &= c_1 \int_K v(x)^2 \phi(\delta(x)^{-2}) dx = c_1 \sigma_v(K).\end{aligned}$$

This proves that  $\gamma_v(E) \geq c_1 \sigma_v(E)$ .  $\square$

Now we can repeat the argument in the proof of [3, Theorem 7.1.3] and conclude that  $\gamma_v = \mathcal{C}_v$  is quasi-additive with respect to  $\{Q_j\}$ .

**Proposition 5.9** *For any Whitney decomposition  $\{Q_j\}$  of  $D$  and any  $v \in \mathcal{S}^c(D)$  satisfying the local scale invariant Harnack inequality with respect to  $\{Q_j\}$ , the Green energy  $\gamma_v$  is locally quasi-additive with respect to  $\{Q_j\}$  for every  $z \in \partial D$ : There exist  $r, c > 0$  such that for every  $z \in \partial D$ ,*

$$c^{-1} \sum_{j \geq 1} \gamma_v(E \cap Q_j) \leq \gamma_v(E) \leq c \sum_{j \geq 1} \gamma_v(E \cap Q_j) \quad \text{for all Borel } E \subset D \cap B(z, r).$$

## 6 Minimal thinness

Throughout this section, we assume that  $\phi$  is a Bernstein function satisfying **(A1)**–**(A6)** and that  $D \subset \mathbb{R}^d$  is either a bounded  $C^{1,1}$  domain, or a  $C^{1,1}$  domain with compact complement or a domain above the graph of a bounded  $C^{1,1}$  function. We assume that the  $C^{1,1}$  characteristics of  $D$  is  $(R, \Lambda)$ .

We start this section by recalling the definition of minimal thinness of a set at a minimal Martin boundary point with respect to the subordinate killed Brownian motion  $Y^D$ .

**Definition 6.1** *Let  $D$  be an open set in  $\mathbb{R}^d$ . A set  $E \subset D$  is said to be minimally thin in  $D$  at  $z \in \partial_m D$  with respect to  $Y^D$  if  $\hat{R}_{M_Y^D(\cdot, z)}^E \neq M_Y^D(\cdot, z)$ .*

For any  $z \in \partial_m D$ , let  $Y^{D,z} = (Y_t^{D,z}, \mathbb{P}_x^z)$  denote the  $M_Y^D(\cdot, z)$ -process, Doob's  $h$ -transform of  $Y^D$  with  $h(\cdot) = M_Y^D(\cdot, z)$ . The lifetime of  $Y^{D,z}$  will be denoted by  $\zeta$ . It is known (see [24]) that  $\lim_{t \uparrow \zeta} Y_t^{D,z} = z$ ,  $\mathbb{P}_x^z$ -a.s. For  $E \subset D$ , let  $T_E := \inf\{t > 0 : Y_t^{D,z} \in E\}$ . It is proved in [14, Satz 2.6] that a set  $E \subset D$  is minimally thin at  $z \in \partial_m D$  if and only if there exists  $x \in D$  such that  $\mathbb{P}_x^z(T_E < \zeta) \neq 1$ .

We assume now that  $z$  is a fixed point in  $\partial D$  and the base point  $x_0$  of the Martin kernel  $M_Y^D$  (cf. (4.3)) satisfies  $x_0 \in D \cap B(z, R)$  and  $R/4 < \delta(x_0) < R$ .

The following criterion for minimal thinness has been proved for a large class of symmetric Lévy processes in [23, Proposition 6.4]. The proof is quite general and it works whenever (1) the cone of excessive functions of the underlying process forms a balayage space, and (2) the inequality in Corollary 4.14 relating the Green function and the Martin kernel of the processes is valid. In particular, the proof works in the present setting. For  $E \subset D$ , define

$$E_n = E \cap \{x \in D : 2^{-n-1} \leq |x - z| < 2^{-n}\}, \quad n \geq 1.$$

**Proposition 6.2** *A set  $E \subset D$  is minimally thin in  $D$  at  $z$  with respect to  $Y^D$  if and only if  $\sum_{n=1}^{\infty} R_{M_Y^D(\cdot, z)}^{E_n}(x_0) < \infty$ .*

Let us fix  $z \in \partial D$ . Define  $v(x) = U^D(x, x_0) \wedge 1$  so that  $v \in \mathcal{S}^c(D)$ . By Theorems 3.1 and 4.5 we see that for  $x$  close to  $z$ ,

$$\frac{M_Y^D(x, z)}{v(x)} \asymp \frac{\phi'(|x - z|^{-2})}{|x - z|^{d+4} \phi(|x - z|^{-2})^2}$$

with a constant depending on  $z$  and  $x_0$ , but not on  $x$ . By using Lemma 2.1(b), we see that there exists a constant  $c_1 > 0$  such that for large  $n$ ,

$$c_1^{-1} \frac{2^{n(d+4)} \phi'(2^{2n})}{\phi(2^{2n})^2} v(x) \leq M_Y^D(x, z) \leq c_1 \frac{2^{(n+1)(d+4)} \phi'(2^{2(n+1)})}{\phi(2^{2(n+1)})^2} v(x), \quad x \in E_n.$$

This implies that

$$c_1^{-1} \frac{2^{n(d+4)} \phi'(2^{2n})}{\phi(2^{2n})^2} R_v^{E_n} \leq R_{M_Y^D(\cdot, z)}^{E_n} \leq c_1 \frac{2^{(n+1)(d+4)} \phi'(2^{2(n+1)})}{\phi(2^{2(n+1)})^2} R_v^{E_n}.$$

In particular,

$$\sum_{n=1}^{\infty} R_{M_Y^D(\cdot, z)}^{E_n}(x_0) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{2^{n(d+4)} \phi'(2^{2n})}{\phi(2^{2n})^2} R_v^{E_n}(x_0) < \infty. \quad (6.1)$$

Since  $\hat{R}_v^{E_n}$  is a potential, there is a measure  $\lambda_n$  (supported by  $\overline{E_n}$ ) charging no polar sets such that  $\hat{R}_v^{E_n} = U^D \lambda_n$ . Also,  $\hat{R}_v^{E_n} = v = U^D(\cdot, x_0)$  on  $\overline{E_n}$  (except for a polar set, and at least for large  $n$ ), hence

$$\begin{aligned} \hat{R}_v^{E_n}(x_0) &= U^D \lambda_n(x_0) = \int_{\overline{E_n}} U^D(x_0, y) \lambda_n(dy) = \int_{\overline{E_n}} v(y) \lambda_n(dy) \\ &= \int_{\overline{E_n}} \hat{R}_v^{E_n}(y) \lambda_n(dy) = \int_D \int_D U^D(x, y) \lambda_n(dy) \lambda_n(dx) = \gamma_v(E_n). \end{aligned}$$

We conclude from (6.1) that

$$\sum_{n=1}^{\infty} R_{M_Y^D(\cdot, z)}^{E_n}(x_0) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{2^{n(d+4)} \phi'(2^{2n})}{\phi(2^{2n})^2} \gamma_v(E_n) < \infty. \quad (6.2)$$

Thus we have proved the following Wiener-type criterion for minimal thinness.

**Corollary 6.3**  *$E \subset D$  is minimally thin in  $D$  at  $z$  with respect to  $Y^D$  if and only if*

$$\sum_{n=1}^{\infty} \frac{2^{n(d+4)} \phi'(2^{2n})}{\phi(2^{2n})^2} \gamma_v(E_n) < \infty.$$

Now we state a version of Aikawa's criterion for minimal thinness.

**Proposition 6.4** *Let  $z \in \partial D$  and  $E \subset D$ , let  $\{Q_j\}$  be a Whitney decomposition of  $D$  and let  $x_j$  denote the center of  $Q_j$ . The following are equivalent:*

(a)  *$E$  is minimally thin at  $z$  with respect to  $Y^D$ ;*

(b)

$$\sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \frac{v^2(x_j) \phi'(\text{dist}(z, Q_j)^{-2})}{\text{dist}(z, Q_j)^{d+4} \phi(\text{dist}(z, Q_j)^{-2})^2} \text{Cap}_D(E \cap Q_j) < \infty;$$

(c)

$$\sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \frac{\text{dist}^2(Q_j, \partial D) \phi'(\text{dist}(z, Q_j)^{-2})}{\text{dist}(z, Q_j)^{d+4} \phi(\text{dist}(z, Q_j)^{-2})^2} \text{Cap}_D(E \cap Q_j) < \infty. \quad (6.3)$$

**Proof.** By using Proposition 5.9, the proof is analogous to the proofs of [23, Proposition 6.6 and Corollary 6.7], cf. also [26, Proposition 4.4], therefore we omit the proof.  $\square$

**Proof of Theorem 1.1:** Assume that  $E$  is minimally thin at  $z \in \partial D$ . By Proposition 6.4, the series (6.3) converges. By Proposition 5.8, the measure

$$\sigma(A) := \int_A \phi(\delta(x)^{-2}) dx, \quad A \subset D,$$

is comparable to the capacity  $\text{Cap}_D$  with respect to the Whitney decomposition  $\{Q_j\}$ . Therefore

$$\text{Cap}_D(E \cap Q_j) \geq c_1 \sigma(E \cap Q_j) = c_1 \int_E \mathbf{1}_{Q_j}(x) \phi(\delta(x)^{-2}) dx.$$

For  $x \in Q_j$  we have that  $\text{dist}^2(Q_j, \partial D) \asymp \delta(x)$  and  $\text{dist}(z, Q_j) \asymp |x - z|$ . Therefore,

$$\begin{aligned} \infty &> \sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \frac{\text{dist}^2(Q_j, \partial D) \phi'(\text{dist}(z, Q_j)^{-2})}{\text{dist}(z, Q_j)^{d+4} \phi(\text{dist}(z, Q_j)^{-2})^2} \text{Cap}_D(E \cap Q_j) \\ &\geq c_2 \sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \int_E \frac{\delta(x)^2 \phi'(|x - z|^{-2})}{|x - z|^{d+4} \phi(|x - z|^{-2})^2} \mathbf{1}_{Q_j}(x) \phi(\delta(x)^{-2}) dx \\ &= c_2 \int_{E \cap B(z, 1)} \frac{\delta(x)^2 \phi(\delta(x)^{-2}) \phi'(|x - z|^{-2})}{|x - z|^{d+4} \phi(|x - z|^{-2})^2} dx. \end{aligned}$$

Conversely, assume that  $E$  is a union of a subfamily of Whitney cubes of  $D$ . Then  $E \cap Q_j$  is either empty or equal to  $Q_j$ . Since  $\text{Cap}_D(Q_j) \asymp \sigma(Q_j) = \int_{Q_j} \phi(\delta(x)^{-2}) dx$ , we can reverse the first inequality in the display above to conclude that

$$\begin{aligned} &\sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \frac{\text{dist}^2(Q_j, \partial D) \phi'(\text{dist}(z, Q_j)^{-2})}{\text{dist}(z, Q_j)^{d+4} \phi(\text{dist}(z, Q_j)^{-2})^2} \text{Cap}_D(E \cap Q_j) \\ &\leq c_3 \int_{E \cap B(z, 1)} \frac{\delta(x)^2 \phi(\delta(x)^{-2}) \phi'(|x - z|^{-2})}{|x - z|^{d+4} \phi(|x - z|^{-2})^2} dx. \end{aligned}$$

$\square$

Theorem 1.1 will be now applied to study minimal thinness of a set below the graph of a Lipschitz function. We start by recalling Burdzy's result, cf. [7, 16]: Let  $f : \mathbb{R}^{d-1} \rightarrow [0, \infty)$  be a Lipschitz function. The set  $A = \{x = (\tilde{x}, x_d) \in \mathbb{H} : 0 < x_d \leq f(\tilde{x})\}$  is minimally thin in  $\mathbb{H}$  with respect to Brownian motion at  $z = 0$  if and only if

$$\int_{\{|\tilde{x}| < 1\}} f(\tilde{x}) |\tilde{x}|^{-d} d\tilde{x} < \infty. \quad (6.4)$$

It is shown recently in [20] that the same criterion for minimal thinness is true for the subordinate Brownian motions studied there. By using Theorem 1.1 one can follow the proof of [20, Theorem 4.4] and show the Burdzy-type criterion for minimal thinness in Proposition 6.5. In the proof we will need the following simple observation: For any  $T > 0$ , we have for  $t \in (0, T]$ ,

$$\int_0^t r^2 \phi(r^{-2}) dr \asymp t^3 \phi(t^{-2}), \quad (6.5)$$

Indeed, since  $r^2 \phi(r^{-2}) \leq t^2 \phi(t^{-2})$  for all  $0 < r \leq t$ , it follows that  $\int_0^t r^2 \phi(r^{-2}) dr \leq t^3 \phi(t^{-2})$ . On the other hand, since  $\phi$  is increasing,  $\int_0^t r^2 \phi(r^{-2}) dr \geq \phi(t^{-2}) \int_0^t r^2 dr = \frac{t^3}{3} \phi(t^{-2})$ .

**Proposition 6.5** *Assume that  $d \geq 3$  and that  $f : \mathbb{R}^{d-1} \rightarrow [0, \infty)$  is a Lipschitz function. Suppose  $D = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > h(\tilde{x})\}$  is the domain above the graph of a bounded  $C^{1,1}$  function  $h$ . Then the set*

$$A := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : h(\tilde{x}) < x_d \leq f(\tilde{x}) + h(\tilde{x})\}$$

*is minimally thin in  $D$  at 0 with respect to  $Y^D$  if and only if*

$$\int_{\{|\tilde{x}| < 1\}} \frac{f(\tilde{x})^3 \phi(f(\tilde{x})^{-2}) \phi'(|\tilde{x}|^{-2})}{|\tilde{x}|^{d+4} \phi(|\tilde{x}|^{-2})^2} d\tilde{x} < \infty. \quad (6.6)$$

**Proof.** Without loss of generality we may assume that  $f(\tilde{0}) = 0$ . We first note that by the Lipschitz continuity of  $f$ , it follows that  $|\tilde{x}| \leq |x| \leq c_1 |\tilde{x}|$  for  $x = (\tilde{x}, x_d) \in A$ . Hence by Fubini's theorem we have

$$\begin{aligned} \int_A \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx &= \int_{|\tilde{x}| < 1} d\tilde{x} \int \mathbf{1}_A(\tilde{x}, x_d) \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx_d \\ &\asymp \int_{|\tilde{x}| < 1} \frac{\phi'(\tilde{x})}{|\tilde{x}|^{d+4} \phi(|\tilde{x}|^{-2})^2} d\tilde{x} \int_0^{f(\tilde{x})} x_d^2 \phi(x_d^{-2}) dx_d \\ &\asymp \int_{|\tilde{x}| < 1} \frac{f(\tilde{x})^3 \phi(f(\tilde{x})^{-2}) \phi'(|\tilde{x}|^{-2})}{|\tilde{x}|^{d+4} \phi(|\tilde{x}|^{-2})^2} d\tilde{x}, \end{aligned} \quad (6.7)$$

where the last asymptotic relation follows from (6.5) with  $T = \sup_{|\tilde{x}| \leq 1} f(\tilde{x})$ . It follows from Theorem 1.1 that if  $A$  is minimally thin in  $D$  at 0, then (6.6) holds true.

For the converse, let  $\{Q_j\}$  be a Whitney decomposition of  $D$  and define  $E = \cup_{Q_j \cap A \neq \emptyset} Q_j$ ; clearly  $A \subset E$ . Let  $Q_j^*$  be the interior of the double of  $Q_j$  and note that  $\{Q_j^*\}$  has bounded multiplicity, say  $N$ . Moreover, if  $Q_j \cap A \neq \emptyset$ , then by the Lipschitz continuity of  $f$  we have  $|Q_j^* \cap A| \asymp |Q_j|$ . Moreover, for  $x \in Q_j^*$  we have  $|x| \asymp \text{dist}(0, Q_j)$ . Therefore

$$\begin{aligned} \int_A \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx &\leq \int_E \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx \\ &= \sum_{Q_j \cap A \neq \emptyset} \int_{Q_j} \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx \\ &\leq c_2 \sum_{Q_j \cap A \neq \emptyset} |Q_j^* \cap A| \frac{\text{dist}^2(Q_j^*, D) \phi(\text{dist}^{-2}(Q_j^*, D)) \phi'(\text{dist}^{-2}(0, Q_j))}{\text{dist}^{d+4}(0, Q_j) \phi(\text{dist}^{-2}(0, Q_j))^2} \\ &\leq c_3 \sum_{Q_j \cap A \neq \emptyset} \int_{Q_j^* \cap A} \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx \leq c_3 N \int_A \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx. \end{aligned} \quad (6.8)$$

If (6.6) holds, then (6.7) and (6.8) imply that

$$\int_E \frac{x_d^2 \phi(x_d^{-2}) \phi'(|x|^{-2})}{|x|^{d+4} \phi(|x|^{-2})^2} dx < \infty.$$

Hence, by Theorem 1.1,  $E$  is minimally thin, and thus  $A$  is also minimally thin.  $\square$

**Remark 6.6** In case  $d \geq 2$  and a bounded  $C^{1,1}$  domain, we can get an analog of Proposition 6.5. Let  $z \in \partial D$  and choose a coordinate system  $CS$  with its origin at  $z$  such that

$$B(z, R) \cap D = \{y = (\tilde{y}, y_d) \text{ in } CS : |y| < R, y_d > h(\tilde{y})\},$$

where  $h$  is a  $C^{1,1}$ -function  $h : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  satisfying  $h(\tilde{0}) = 0$ . Let  $f : \mathbb{R}^{d-1} \rightarrow [0, \infty)$  be a Lipschitz function and

$$A := \{x = (\tilde{x}, x_d) \in D : |x| < R, h(\tilde{x}) < x_d \leq f(\tilde{x}) + h(\tilde{x})\}.$$

Then the set is minimally thin in  $D$  at  $z \in \partial D$  with respect to  $Y^D$  if and only if (6.6) holds true.

## 7 Examples

In this section we assume  $D$  is either a bounded  $C^{1,1}$  domain in  $\mathbb{R}^d$  or a half-space. We first compare criteria for minimal thinness for three processes in  $D$  related to the isotropic  $\alpha$ -stable process. The first process is the killed isotropic  $\alpha$ -stable process  $X^D$ ,  $0 < \alpha < 2$ , that is a killed subordinate Brownian motion  $X_t = W_{S_t}$  where  $(S_t)_{t \geq 0}$  is an  $(\alpha/2)$ -stable subordinator. The corresponding Laplace exponent is the function  $\phi(\lambda) = \lambda^{\alpha/2}$ . The second process is the subordinate killed Brownian motion  $Y_t^D = W_{S_t}^D$  with the same  $(\alpha/2)$ -stable subordinator. The third process is the censored  $\alpha$ -stable process  $Z^D$ . The process  $Z^D$  is a symmetric Markov process with Dirichlet form given by

$$\mathcal{C}(v, v) = \int_D \int_D (v(y) - v(x))^2 j(y - x) dy dx,$$

where  $j(x)$  is the density of the Lévy measure of the isotropic  $\alpha$ -stable process. The censored stable process was introduced and studied in [6]. When  $\alpha \in (1, 2)$ ,  $Z^D$  is transient and converges to the boundary of  $D$  at its lifetime.

Hardy's inequality for the Dirichlet form of  $Z^D$  was obtained in [10, 13]. Let  $G_Z^D$  be the Green function of  $Z^D$ . If  $D$  is a bounded  $C^{1,1}$  domain, sharp two-sided estimates on  $G_Z^D$  were obtained in [8]. If  $D$  is a half-space, say the upper half-space, then it follows from [6] that the censored  $\alpha$ -stable process in  $D$  satisfies the following scaling property: for any  $c > 0$ , if  $(Z_t^D)_{t \geq 0}$  is a censored  $\alpha$ -stable process in  $D$  starting from  $x \in D$ , then  $(cZ_{t/c^\alpha}^D)_{t \geq 0}$  is a censored  $\alpha$ -stable process in  $D$  starting from  $cx$ . Thus the transition density  $p_Z^D(t, x, y)$  of  $Z^D$  satisfies the following relation:

$$p_Z^D(t, x, y) = t^{-d/\alpha} p_Z^D(1, t^{-1/\alpha} x, t^{-1/\alpha} y), \quad t > 0, x, y \in D.$$

Now using the short-time heat kernel estimates in [9] we immediately arrive at the following global estimates:

$$p_Z^D(t, x, y) \asymp t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x - y|}\right)^{d+\alpha} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}, \quad \text{on } (0, \infty) \times D \times D.$$



Using the above estimates, one can easily get sharp two-sided estimates on  $G_D$  from which one can easily show that

$$\lim_{x \ni D \rightarrow \infty} \frac{G_Z^D(x, y)}{G_Z^D(z, y)} = 0.$$

Sharp two-sided estimates on  $G_Z^D$  give sharp two-sided estimates on the Martin kernel of  $Z^D$ . The arguments in [8] imply that the finite part of the minimal Martin boundary of  $D$  with respect to  $Z^D$  and the finite part of the Martin boundary of  $D$  with respect to  $Z^D$  both coincide with the Euclidean boundary  $\partial D$  of  $D$ .

Based on these results, one can follow the proof of [26, Proposition 4.4] (which is an analog of Proposition 6.4) line by line and see that the same results also hold when  $D$  is a half-space. Therefore the following holds.

**Proposition 7.1** *Let  $\alpha \in (1, 2)$  and  $d \geq 2$ . Let  $D$  be either a bounded  $C^{1,1}$  domain in  $\mathbb{R}^d$  or a half-space,  $z \in \partial D$ ,  $E \subset D$ , and let  $x_j$  denote the center of  $Q_j$ . Let  $x_0 \in D$  be fixed,  $\text{Cap}^D$  be the capacity with respect to  $Z^D$  and  $v(x) = G_Z^D(x, x_0) \wedge 1$ . The following are equivalent:*

(a)  *$E$  is minimally thin at  $z$ ;*

(b)

$$\sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \text{dist}(z, Q_j)^{-d-\alpha+2} v(x_j)^2 \text{Cap}^D(E \cap Q_j) < \infty; \quad (7.1)$$

(c)

$$\sum_{j: Q_j \cap B(z, 1) \neq \emptyset} \frac{\text{dist}(Q_j, \partial D)^{2(\alpha-1)}}{\text{dist}(z, Q_j)^{d+\alpha-2}} \text{Cap}^D(E \cap Q_j) < \infty. \quad (7.2)$$

It is shown in [26] that the measure  $\sigma(A) := \int_A \delta(x)^{-\alpha} dx$  is comparable to  $\text{Cap}^D$  with respect to the Whitney decomposition. Further, it follows from [8, Theorem 1.1] that  $v(x_j) \asymp \text{dist}(Q_j, \partial D) \asymp \delta(x)^{2(\alpha-1)}$  for all  $x \in Q_j$ . With this in hand one can use the argument in the proof of Theorem 1.1 to prove the following criterion for minimal thinness with respect to the censored  $\alpha$ -stable process.

**Theorem 7.2** *Assume that  $\alpha \in (1, 2)$ . Let  $D$  be either a bounded  $C^{1,1}$  domain in  $\mathbb{R}^d$  or a half-space,  $d \geq 2$ , and let  $E$  be a Borel subset of  $D$ .*

(1) *If  $E$  is minimally thin in  $D$  at  $z \in \partial D$  with respect to  $Z^D$ , then*

$$\int_{E \cap B(z, 1)} \frac{\delta(x)^{\alpha-2}}{|x - z|^{d+\alpha-2}} dx < \infty.$$

(2) *Conversely, if  $E$  is the union of a subfamily of Whitney cubes of  $D$  and is not minimally thin in  $D$  at  $z \in \partial D$  with respect to  $Y^D$ , then*

$$\int_{E \cap B(z, 1)} \frac{\delta(x)^{\alpha-2}}{|x - z|^{d+\alpha-2}} dx = \infty.$$

Note that for  $X^D$  the integral in the criterion for minimal thinness is

$$\int_{E \cap B(z,1)} \frac{1}{|x-z|^d} dx ,$$

while for  $Y^D$  the corresponding integral becomes

$$\int_{E \cap B(z,1)} \frac{\delta(x)^{2-\alpha}}{|x-z|^{d+2-\alpha}} dx .$$

**Corollary 7.3** *Let  $D$  be either a bounded  $C^{1,1}$  domain in  $\mathbb{R}^d$  with  $d \geq 2$  or a half-space with  $d \geq 3$ . Let  $E$  be the union of a subfamily of Whitney cubes of  $D$  and  $z \in \partial D$ .*

*(i) Let  $1 < \alpha < 2$ . If  $E$  is minimally thin at  $z$  with respect to  $Z^D$ , then it is minimally thin at  $z$  with respect to  $X^D$ .*

*(ii) Let  $0 < \alpha < 2$ . If  $E$  is minimally thin at  $z$  with respect to  $X^D$ , then it is minimally thin at  $z$  with respect to  $Y^D$ .*

*(iii) Let  $1 < \alpha_1 \leq \alpha_2 < 2$ . If  $E$  is minimally thin at  $z$  with respect to the  $\alpha_1$ -stable censored process, then it is minimally thin at  $z$  with respect to the  $\alpha_2$ -stable censored process.*

*(iv) Let  $0 < \alpha_1 \leq \alpha_2 < 2$ . If  $E$  is minimally thin at  $z$  with respect to  $Y^D$  with index  $\alpha_2$ , then it is minimally thin at  $z$  with respect to  $Y^D$  with index  $\alpha_1$ .*

**Proof.** All statements follow easily from criteria in Theorems 1.1 and 7.2 together with the observation that since  $\delta(x) \leq |x-z|$ ,

$$\left( \frac{\delta(x)}{|x-z|} \right)^{2-\alpha} \leq 1 \leq \left( \frac{\delta(x)}{|x-z|} \right)^{\alpha-2} .$$

□

A criterion for minimal thinness of a set below the graph of a Lipschitz function with respect to the censored stable process is given in the following result which can be proved in the same way as Proposition 6.5.

**Proposition 7.4** *Let  $\alpha \in (1, 2)$ . Assume that  $f : \mathbb{R}^{d-1} \rightarrow [0, \infty)$  is a Lipschitz function. Suppose that  $D = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : 0 < x_d\}$ . Then the set*

$$A := \{x = (\tilde{x}, x_d) \in D : 0 < x_d \leq f(\tilde{x})\}$$

*is minimally thin in  $D$  at 0 with respect to  $Z^D$  if and only if*

$$\int_{\{|\tilde{x}| < 1\}} \frac{f(\tilde{x})^{\alpha-1}}{|\tilde{x}|^{d+\alpha-2}} d\tilde{x} < \infty . \quad (7.3)$$

In case of  $X^D$ , the criterion reads

$$\int_{\{|\tilde{x}| < 1\}} \frac{f(\tilde{x})}{|\tilde{x}|^d} d\tilde{x} < \infty , \quad (7.4)$$

while for  $Y^D$  with  $d \geq 3$ , (6.6) becomes

$$\int_{\{|\tilde{x}| < 1\}} \frac{f(\tilde{x})^{3-\alpha}}{|\tilde{x}|^{d+2-\alpha}} d\tilde{x} < \infty . \quad (7.5)$$

**Example 7.5** Let  $d \geq 3$  and  $D = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : 0 < x_d\}$ ,  $f : \mathbb{R}^{d-1} \rightarrow [0, \infty)$  a Lipschitz function and put  $A := \{x = (\tilde{x}, x_d) \in D : 0 < x_d \leq f(\tilde{x})\}$ .

(1) If  $f(\tilde{x}) = |\tilde{x}|^\gamma$  with  $\gamma \geq 1$ , then an easy calculation shows that all three integrals in (7.3)-(7.5) are finite if and only if  $\gamma > 1$ . Thus, for all three processes,  $A$  is minimally thin at  $z = 0$  if and only if  $\gamma > 1$ .

(2) Let  $f(\tilde{x}) = |\tilde{x}|(\log(1/|\tilde{x}|))^{-\beta}$ ,  $\beta \geq 0$ . Then  $f$  is Lipschitz. By use of (7.3)-(7.5) it follows easily that  $A$  is minimally thin at  $z = 0$

with respect to  $Z^D$  if and only if  $\beta > \frac{1}{\alpha - 1}$ ,

with respect to  $X^D$  if and only if  $\beta > 1$ ,

with respect to  $Y^D$  if and only if  $\beta > \frac{1}{3 - \alpha}$ .

Since  $1 < 1/(3 - \alpha)$  for  $\alpha \in (0, 2)$  and  $1 < 1/(\alpha - 1)$  for  $\alpha \in (1, 2)$  this is in accordance with Corollary 7.3. By choosing  $\beta$  and  $\alpha$  appropriately, we conclude that none of the converse in Corollary 7.3 holds true.

We conclude this paper with an example about minimal thinness with respect to subordinate killed Brownian motion in the half-space via geometric stable subordinators. We define  $L_1(\lambda) = \log \lambda$ , and for  $n \geq 2$  and  $\lambda > 0$  large enough,  $L_n(\lambda) = L_1(L_{n-1}(\lambda))$ . Applying Proposition 6.5, we can easily check the following.

**Example 7.6** Let  $d \geq 3$  and  $\alpha \in (0, 1]$ . Suppose that  $D = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : 0 < x_d\}$  and  $Y^D$  is the subordinate killed Brownian motion in  $D$  via a subordinator with Laplace exponent  $\log(1 + \lambda^\alpha)$ . Assume that  $f : \mathbb{R}^{d-1} \rightarrow [0, \infty)$  a Lipschitz function and define  $A := \{x = (\tilde{x}, x_d) \in D : 0 < x_d \leq f(\tilde{x})\}$ .

(1) Let  $f(\tilde{x}) = |\tilde{x}|(L_1(1/|\tilde{x}|))^{-\beta}$  with  $\beta \geq 0$ . Then  $A$  is minimally thin at  $z = 0$  with respect to  $Y^D$  if and only if  $\beta > 0$ .

(2) Let  $n \geq 2$  and  $f(\tilde{x}) = |\tilde{x}|(L_2(1/|\tilde{x}|) \cdots L_n(1/|\tilde{x}|))^{-1/3} (L_{n+1}(1/|\tilde{x}|))^{-\beta}$  with  $\beta \geq 0$ . Then  $A$  is minimally thin at  $z = 0$  with respect to  $Y^D$  if and only if  $\beta > 1/3$ .

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